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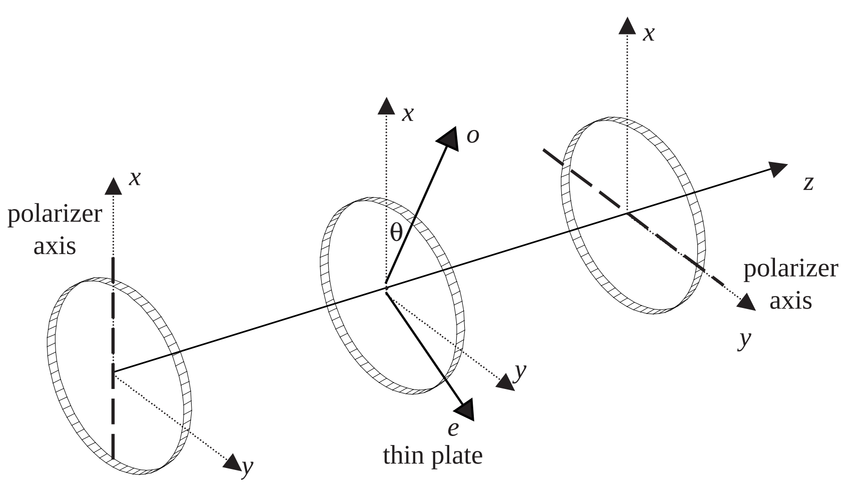
2003 UNIVERSITY PRIZE EXAMINATION
CANADIAN ASSOCIATION OF PHYSICISTS

Wedenesday, February 5, 2003

Instructions: Please write your solutions to different problems on separate booklets (and write your name on each), as they will be marked by different people. It is practically impossible to answer all the problems in the time given. Do not feel frustrated by that. Rather, try to select the problems that will best match your abilities.

Question 1: Polarisation of light (10 points)

A beam of light passes through a first polarizer, then a birefringent thin plate and then a second polarizer, at normal incidence (see figure). The thickness and the refractive indices of the plate are such that a retardation phase φ is introduced between the electric field components along the extraordinary and ordinary axes, respectively.



a) If the ordinary axis of the plate makes an angle θ with respect to the first polarizer and if the second polarizer is orthogonal to the first one, find the expression of the intensity after the second polarizer as a function of the intensity I_0 measured after the first polarizer.

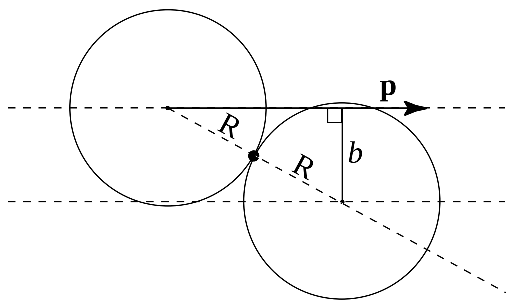
b) What is the thickness of the birefringent plate and the angle θ necessary for the intensity after the second polarizer to be $I_0/2$. The wavelength is$\lambda_0 = 600$nm and the ordinary and extraordinary indices are $n_o = 1.544$ and $n_e = 1.553$. Note: the solution for θ and φ is not unique, however a simple solution can be found for which numerical values are obtained easily.

c) If a second retardation plate, identical to the first one, is inserted between the two polarizers, explain how to obtain a configuration such that the intensity after the second polarizer is equal to I_0. Justify your answer.

Question 2: Collision of hard spheres (10 points)

A hard sphere of radius R and momentum $\mathbf{p}$ hits a second sphere, identical to the first but a rest (see figure below). The only force in action is a contact force exerted by one sphere upon the other, directed along the straight line that goes through the centers of the two spheres. This force is very intense, acts during a very short time and gives the sphere at rest a momentum $\mathbf{q}$. The momentum of the first sphere following the collision is $\mathbf{p}'$. The distance between the center of the second sphere and a line parallel to $\mathbf{p}$ going through the center of the first sphere (the impact parameter), is denoted b. Assume that the collision is elastic.

Express, as a function of the parameters of the problem (b, R and $\mathbf{p}$), the momenta $\mathbf{p}'$ and $\mathbf{q}$ of the two spheres after the collision (in size and direction).



Question 3: Superconducting sphere in a magnetic field (15 points)

A type-I superconductor has the property of excluding any magnetic field from its interior. Thus, if a superconducting object is placed in an external magnetic field, a current density $\mathbf{K}$ (current per unit length) is induced at the surface of the object, such as to cancel the external magnetic field inside the object. Consider a superconducting sphere of radius a, subjected to a uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. The goal of the problem is to calculate the net magnetic field everywhere outside the sphere and the associated induced surface current density $\mathbf{K}$. The problem is static (nothing changes as a function of time).

a) Knowing that the only current source is on the surface of the sphere, explain why the field $\mathbf{B}$ outside the sphere may be written as the gradient of a function outside the sphere : $\mathbf{B} = -\nabla\Phi$, and why Φ obeys the Laplace equation $\nabla^2\Phi = 0$.

b) Explain why the component of $\mathbf{B}$ normal to the sphere must vanish at the surface.

c) Find Φ everywhere outside the sphere.

d) Show that the component of $\mathbf{B}$ parallel to the surface is discontinuous at the surface, and that its value just above the surface is proportional to the current density $\mathbf{K}$. Then proceed to calculate the current density $\mathbf{K}$.

Note : The general solution to Laplace's equation in spherical coordinates, for a problem with azimuthal symmetry, is

$$\Phi(r,\theta) = \sum_l \left(A_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

where A_l et C_l are constants and where $P_l(x)$ is the Legendre polynomial of order l. In particular,

$$P_0(x) = 1 \quad , \quad P_1(x) = x \quad , \quad P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}$$

and higher polynomials may be calculated from the recursion relation :

$$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x)$$

Maxwell's equations, in the CGS system, are:

$$\nabla\cdot\mathbf{E} = 4\pi\rho \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\wedge\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0 \qquad \nabla\wedge\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$$

The gradient operator in spherical coordinates is :

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

Question 4: Landau levels (15 points)

Consider an electron of charge $e < 0$ and mass m subjected to a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ and confined to the xy plane (no motion along the z direction possible). The Hamiltonian takes the following form :

$$H = \frac{1}{2m}\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2 .$$

We will adopt the Landau gauge : $\mathbf{A} = (0, Bx, 0)$.

a) Show that the possible electron energies are those of a one-dimensional harmonic oscillator of frequency $\omega_c = eB/mc$, the cyclotron frequency.

b) If we call φ_n the normalized eigenfunctions of the harmonic oscillator, what are the eigenfunctions of the hamiltonian in the present case? What are the quantum numbers that label the eigenfunctions?

c) What is the degeneracy of each energy level in a system of size $L_x \times L_y$? Use periodic boundary conditions along the y axis.

d) We now add a uniform electric field $\mathbf{E} = E\hat{\mathbf{x}}$. After a suitable modification of your solution, show that the electric field lifts the degeneracy of the energy levels. What are the new values E' of the possible energies with this perturbed hamiltonian H'?

Question 5: Linear Stark Effect (15 points)

Consider a hydrogen atom in a constant electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ oriented along the z direction (neglect the electron spin and assume that E_0 is weak compared to the Coulomb field of the atom). The atom is coupled to the electric field by the interaction term

$$W = -\mathbf{D}\cdot\mathbf{E}$$

where $\mathbf{D} = q\mathbf{R}$ is the dipole moment and q is the electric charge.

a) In the absence of field, the hydrogen atom is prepared in the first excited state of energy $E_n = -E_I/n^2$ where $n = 2$ and E_I is the ionisation energy of the atom. Show that within the $n = 2$ Hilbert subspace $\mathcal{E}_2$, the electric field only couples the 2s ($\ell = 0$) and 2p$_z$ ($\ell = 1, m = 0$) states. In the matrix representation, show that the total Hamiltonian $H = H_0 + W$ in $\mathcal{E}_2$ reduces to

$$(H)_{\mathcal{E}_2} = \begin{pmatrix} E_2 & 3qa_0E_0 \\ 3qa_0E_0 & E_2 \end{pmatrix},$$

where a_0 is the Bohr radius and H_0 is the Hamiltonian in the absence of electric field.

b) Show that the electric field induces a dipolar moment.

c) At time $t = 0$, the system is prepared in the eigenstate 2s of H_0. What is the probability at time $t > 0$ to find the atom in the state 2p$_z$?

Note: Eigenfunctions $\psi_{n,\ell,m}(r, \theta, \varphi)$ of the hydrogen atom:

$$\begin{aligned}
\psi_{2,0,0}(r,\theta,\varphi) &= \frac{1}{\sqrt{8\pi a_0^3}}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \\
\psi_{2,1,\pm1}(r,\theta,\varphi) &= \mp\frac{1}{8\sqrt{\pi a_0^3}}\frac{r}{a_0} e^{-r/2a_0}\sin\theta\, e^{\pm i\varphi} \\
\psi_{2,1,0}(r,\theta,\varphi) &= \frac{1}{4\sqrt{2\pi a_0^3}}\frac{r}{a_0} e^{-r/2a_0}\sin\theta
\end{aligned}$$

Useful integral:

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

Question 6: Relativistic retardation of clocks (10 points)

The space shuttle follows a circular orbit of radius $r = R_\oplus + h$ around the Earth ($R_\oplus$ is the Earth's radius [6378 km] and h the shuttle's altitude [$\sim$ 200 km]).

a) Suppose that an extremely accurate atomic clock is carried by the shuttle. Because of the shuttle's speed, this clock lags behind its twin, which remained on the ground. Express this lag (defined as the fractional change $\Delta T/T$ in the "period" of the clock) as a function of r, $M_\oplus$, G (the gravitationnal constant) and c (the speed of light). Neglect the Earth's rotation. Make the necessary approximations, knowing that the shuttle's speed is small compared to c.

b) The Earth's gravitational field can also modify the flow of time, according to Einstein's general relativity. This retardation may be related to the variation of the wavelength λ of a light wave propagating in the vicinity of the Earth. One shows that the combination

$$\frac{\lambda}{\sqrt{1 - \frac{2r_g}{r}}}$$

is constant during propagation, where

$$r_g \equiv \frac{GM_\oplus}{c^2}$$

is the gravitational radius of the nearby object (here, the Earth). Explain how to calculate the retardation of a clock located on the ground compared to an identical clock located at an altitude h, due to this effect.

c) For the clock aboard the shuttle, this last effect is opposite to the one calculated in (a). Do you think it is more important, less important, or of equal importance? In other words, which one of the two clocks will really lag behind once they are brought back together? Justify your statements quantitatively. You may neglect the variation of g between the ground and the shuttle.

d) Would this be different for an atomic clock in a satellite with an orbiting radius of 20 000 km, like those of the GPS (*Global Positioning System*)?

Question 7: Partition function and thermodynamics of the Van der Waals gas (10 points)

The partition function of a Van der Waals gas may be written as

$$Z(T,V) = \frac{1}{N!h^{3N}} (2\pi m k_B T)^{3N/2} \left[(V - Nb) \exp\left(\frac{a}{k_B T} \frac{N}{V} \right) \right]^N$$

where T is the absolute temperature, N the number of molecules in the gas, V the volume, m the mass of each molecule, k_B is Boltzmann's constant, h is Planck's constant and a, b are positive constants.

Remark: each of the sub-questions below, from (a) to (e), can be answered in any order.

a) Explain in a few words the origin of the term $(2\pi m k_B T)^{3N/2}$.

b) Find the relation between T and V in an adiabatic process. (recall $S = -(\partial F/\partial T)_V$ with $F = -k_B T \ln Z$. Stirling's formula : $\ln N! \approx N \ln N - N$.)

c) Find the work done by a Van der Waals gas that changes from a volume V_0 to a volume V_1 at constant temperature. (Recall $p = -(\partial F/\partial V)_T$).

d) Consider and isolated Van der Waals gas (constant energy) enclosed on one side of a box. A hole is opened between the two sides of the box. Without adding or removing energy, one lets the gas occupy the two sides of the box such that the volume occupied by the gas changes from V_0 to V_1. Compute the corresponding change in temperature. Does the system cool down or warm up?

e) Find an expression for the fluctuations of the total energy at equilibrium.

Question 8: Evaporation of a planetary atmosphere: Jeans model (15 points)

The purpose of this problem is to study the evaporation of an isothermal planetary atmosphere obeying the Maxwell distribution. We have adopted the following notation: dJ is the total number of particles crossing an elemental area ds per unit time; this flux dJ is related to the flux density j (flux per unit area) through $j = dJ/ds$; n is the particle concentration in the atmosphere (i.e. the number of particles per unit volume).

In the case of particles moving with velocity $\mathbf{v}$ and crossing an elemental area $d\mathbf{s} = ds\,\hat{\mathbf{r}}$ ($\hat{\mathbf{r}}$ is the unit vector perpendicular to ds), one has $dJ = n\,\mathbf{v} \cdot d\mathbf{s}$. In what follows, $\mathbf{j} = n\,\mathbf{v}$ is the flux density vector.

a) Show that the elemental flux density of particles having momentum $\mathbf{p} = m\mathbf{v}$ within the momentum space volume element $d\Omega = dp_x dp_y dp_z$ has the following distribution:

$$d\mathbf{j} = \frac{\mathbf{p}}{m} \frac{n}{(2\pi m k T)^{3/2}} e^{-p^2/2mkT} d\Omega,$$

where m is the mass of a particle, k is the Boltzmann constant and T is the temperature of the atmosphere which is considered to be a ideal gas. Recall :

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-x^2/2\sigma^2} = \sqrt{2\pi\sigma^2}$$

b) Show that the atmospheric particles *moving away* from the planet and having a kinetic energy larger than a given threshold A give rise to

$$j = \frac{n}{2\sqrt{\pi}}\sqrt{\frac{2kT}{m}}e^{-A/kT}(1+\frac{A}{kT}).$$

c) Give the expression for the minimal velocity required of a particle to overcome the gravitational attraction of a planet of mass M and radius R. This escape velocity will be referred to as v_{lib}.

d) Assuming the planet's atmosphere is homogeneous, *isothermal*, in hydrostatic equilibrium, and obeys the ideal gas law ($P = \rho kT/m$ with $\rho = nm$), show that one can define an effective thickness for the atmosphere $H_{\text{eff}} = kT/(mg)$ (g is the gravitational acceleration on the planet's surface).

The exobase is the level of the planetary atmosphere at which the mean free path of the particles is equal to H_{eff}. The region above is called the exosphere. In the exosphere, the atmosphere is not dense enough to allow retention of the particles (via collisions) whose velocity is greater than the escape velocity.

e) Presuming H_{eff} is small compared to the planet's radius (i.e. the area covered by the atmosphere at any altitude is approximately equal to the planet's surface area), show that the atmosphere evaporates as a function of time t according to a relation of the type $n(t) = n_0 \times e^{-t/\tau}$, where n_0 is the initial concentration of the particles and τ is a characteristic time equal to:

$$\tau = \frac{2H_{\text{eff}}}{\sqrt{\frac{2kT}{m\pi}}}e^{mv_{\text{lib}}^2/2kT}\left(1+\frac{mv_{\text{lib}}^2}{2kT}\right)^{-1} = \frac{1}{g}\sqrt{\frac{2\pi kT}{m}}e^{mv_{\text{lib}}^2/2kT}\left(1+\frac{mv_{\text{lib}}^2}{2kT}\right)^{-1}.$$

f) Knowing that the solar system formed some 4.5 billion years ago, give an estimate and compare the present concentrations of nitrogen molecules within the exospheres of the Earth and the Moon. It should be verified that H_{eff} is indeed small compared to the respective radius of the bodies involved. One can assume that the initial concentration, n_0, was the same for the Earth and the Moon.

One can use the following values: the gravitational constant $G = 6.67 \times 10^{-11}$ SI, the radius of the Earth $R_T = 6378$ km, the radius of the Moon $R_L = 1738$ km, the mass of the Earth $M_T = 6.0 \times 10^{24}$ kg and the mass of the Moon $M_L = 7.3 \times 10^{22}$ kg, the mass of a nitrogen molecule $m = 4.7 \times 10^{-26}$ kg, $T = 900$ K and $k = 1.38 \times 10^{-23}$ J K^{-1}.