Canadian Association of Physicists Undergraduate Prize Exam, 2009

Instructions

Duration: three hours

Calculators may be used.

There are a total of 10 questions.

Write each solution on a separate page. If more than one page is required for any question, then those pages should be stapled together separate from other questions.

Please write the number of the question, your name and the name of your university clearly on the first page of each answer.

Each question has the same value. You are not expected to attempt all the questions. Relax and attempt the questions on material that you are most familiar with or those questions that look the most interesting to you.

Completed exam papers should be sent by departmental organizers to: Prof. Ian Affleck, Dept. of Physics and Astronomy, University of British Columbia, Vancouver, B.C., V6T 1Z1.

This year's exam was prepared at University of British Columbia by Ian Affleck, with assistance from: Mona Berciu, Marcel Franz, Brett Gladman, David Jones, Joanna Karczmarek, Steve Plotkin, Bill Unruh, Mark Van Raamsdonk, Silke Weinfurtner and Fei Zhou. The French translation was done by David Sénéchal, Université de Sherbrooke.

Problem 1

Find the period of small oscillations for the planar pendulum shown in Fig. 1. The wire has a total length $\pi R < L < 2\pi R$, and is nailed to the fixed frame of radius R at point P. Assume that the portion of the wire not touching the frame remains straight, as shown and ignore friction.

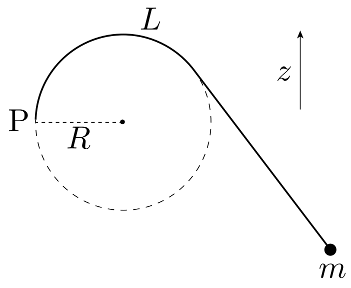

FIG. 1: planar pendelum - z is the vertical direction

Problem 2

A block of right-angle triangular cross-section and mass M is sliding on a frictionless floor while a homogeneous cylinder of mass m and radius R is rolling without slipping down a side of the block oriented at an angle α from the horizontal. (See figure.) Assume that initially both block and cylinder are at rest with the point of contact of the cylinder and the block at a height $H \gg R$ above the floor.

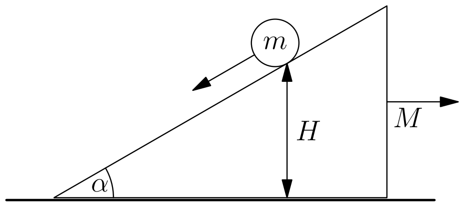

FIG. 2: block and cylinder on floor

a) Find the equations of motion.

b) After how much time will the cylinder reach the floor?

Problem 3

A particle interacts with a background scalar field which affects its inertial mass. In a particular inertial frame, the scalar field appears static and the particle's rest mass depends on its position x via $m(x) = m_0 e^{-\alpha x}$, $\alpha > 0$. At time $t = 0$, the particle is at rest at $x = 0$. At later times it is moving to the right with a non-zero velocity. Compute its subsequent trajectory, using relativistic mechanics.

Problem 4

Optical traps that are made up of interfering laser beams have recently been used to capture and confine ultra cold atoms. Near the centre of an optical trap, laser beams produce an effective electric field of the form

$$\vec{E}(x) = E_0(1 - x^2/x_0^2)\hat{e}_z \tag{1}$$

where typically, $E_0 = 5,000$ V/m, $x_0 = 5\mu$m and x is the distance from the centre of the trap. A rubidium atom (^{87}Rb) moving along the x-direction with 0.1 mm/s speed is located at $x = 0$ when the trap is turned on. According to the periodical table, this isotope has 37 protons, 50 neutrons and one 5s electron; the atomic radius is about 2.5A.

a) Treating the atom as a point nucleus surrounded by a neutralizing uniformly charged sphere, calculate its polarizability (electric dipole moment divided by electric field).

b) Describe quantitatively the motion of this rubidium atom after the trap is turned on. (Find the time and length scale of the motion of this atom in the trap; specify the assumptions used in the problem.)

c) In order to be captured by the trap, what is the maximal speed a rubidium atom can have when the trap is turned on?

Problem 5

Consider a piece of glass (with index of refraction $n = 1.5$) with normally incident light at wavelength 530 nm.

a) Design a single dielectric layer which could be applied to the glass that acts as an anti-reflection coating. In other words, what is the thickness (d) and index of refraction (n_f) of a single dielectric layer that would prevent any of the light from reflecting back to the left, towards the source? Assume the glass is infinitely thick. (To solve this problem you will need to consider the multiple reflections from the air-film interface and the film-glass interface. You may assume all media are lossless nonmagnetic dielectrics.) (See figure.)

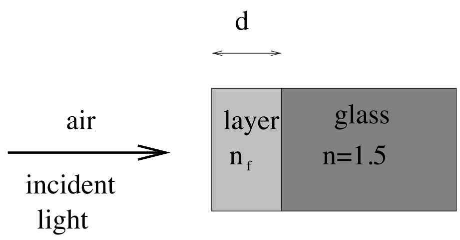

FIG. 3: anti-reflection coating

b) Based on your answer to part a) explain why the anti-reflection coating on your eyeglasses looks purple when viewed at an angle.

Problem 6

A particular quantum system can be modeled by the illustrated configuration of balls and springs, where m and k are the masses of the balls and the spring constants of the springs and the balls are constrained to move in the x-direction only.

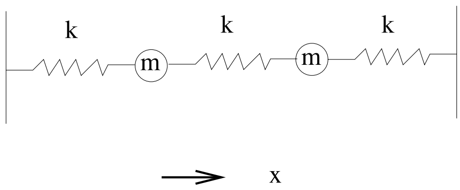

FIG. 4: quantum balls and springs. The outer ends of the springs are fixed.

a) Determine the spectrum of energy eigenvalues of this system, assuming that the classical energy in the equilibrium configuration shown is zero.

b) What is the expected value of $(l - l_0)^2$ in the ground state where l and l_0 are the distance between the balls and the equilibrium distance between the balls respectively?

Problem 7

A protein may be modeled as a particular configuration of a collapsed polymer globule (see figure). Let this particular polymer consist of N residues (or beads) connected by $N - 1$ bonds. Let there be ν orientational states per bond.

a) If the reorientation time between bond states is 10^{-12} s, estimate the time it would take for the polymer to find the particular "native" conformation of the protein by random search through conformations. For this part of the problem, let $\nu = 10$, and $N = 101$. Is your estimate reasonable? (i.e. is this a reasonable time scale for a protein to fold?)

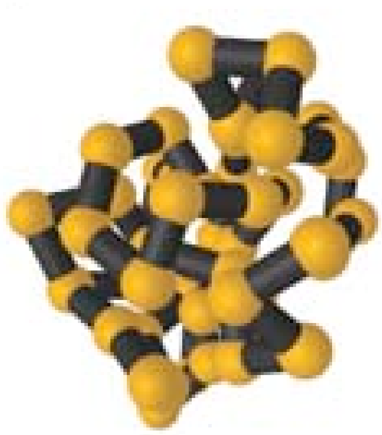

FIG. 5: collapsed polymer globule

b) In the collapsed conformation, let each bead have z nearest neighbors that it interacts with. Furthermore let the energy of each interaction be randomly distributed according to the distribution shown in the figure. In the limit that

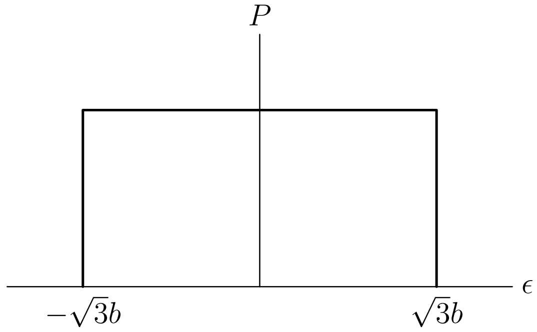

FIG. 6: probability of energy being ϵ

the polymer has a large number of interacting monomers, what is the probability (in terms of N, b, z and any other variables) that a particular conformation has energy E?

c) Find the average energy in the system at temperature T (you can ignore any non-extensive corrections).

d) Find the entropy at temperature T (ignoring non-extensive corrections) and show that there is temperature where the system runs out of entropy and thus freezes into one state. What is that temperature? (This is not protein folding, but it is more like a glass transition in a nanoscopic system.)

Problem 8

Any solid state crystal can be viewed as a Bravais lattice of equivalent points with a basis of atoms associated with each of these points. The Wigner-Seitz unit cell of a crystal is the most symmetric possible primitive unit cell. By displacing it by all Bravais lattice vectors, all of space is filled once. The reciprocal lattice is the Bravais lattice consisting of the set of vectors, $\vec{G}$, satisfying the condition:

$$e^{i\vec{G}\cdot\vec{R}} = 1 \tag{2}$$

for each vector $\vec{R}$ in the direct Bravais lattice. The first Brillouin zone is the Wigner-Seitz unit cell of the reciprocal lattice.

When a crystal is irradiated with X-rays, the scattering intensity, for some change in wave-vector $\Delta\vec{k}$ for the radiation, vanishes due to destructive interference, unless $\Delta\vec{k} = \vec{G}$ for some reciprocal lattice vector $\vec{G}$. When this condition is satisfied, the scattering intensity is proportional to the square of the structure factor, $S_{\vec{G}}$ where:

$$S_{\vec{G}} = \sum_a f_a e^{i\vec{G}\cdot\vec{r}_a}. \tag{3}$$

Here the sum is over the positions, $\vec{r}_a$ of all atoms in one unit cell of the crystal and the f_a are the form factors, or scattering amplitudes for each atom. (In general, the f_a may depend on the change in wave-vector $\vec{G}$, but we ignore this here for simplicity. This is sometimes a reasonable approximation for the first few reciprocal lattice wave-vectors of smallest magnitude.)

Consider a simplified copper-oxygen crystal. On each $x-y$ plane, Cu atoms form a square lattice with spacing a while O atoms occupy sites midway between each neighbouring pair of Cu atoms. These planes simply repeat in the z direction, with separation c. (See figure)

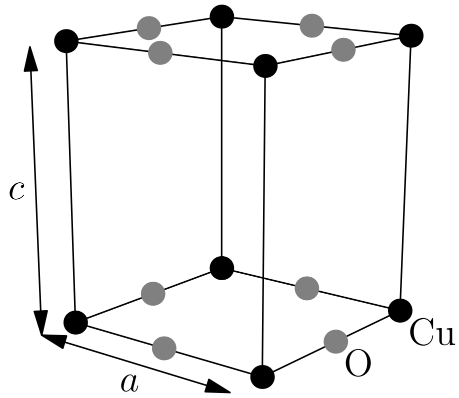

FIG. 7: A simplified Cu-O crystal.

a) Sketch the Wigner-Seitz unit cell of this crystal, projected onto the x-y plane, showing the location of the atoms. Find the volume of the Wigner-Seitz unit cell.

b) Find the reciprocal lattice vectors, $\vec{G}$.

c) Assuming that the atomic form factors are f_{Cu} and f_O for the two types of atoms (copper and oxygen respectively) calculate the structure factor, $S_{\vec{G}}$.

d) Based on the above, what are the ratios of heights of the peaks observed in X-ray scattering?

Problem 9

The 'dwarf planet' Eris is currently 96.7 AU (1.45×10^{13} m) from the Sun (more than three times further than Neptune). While the Earth at 1 AU receives 1340 W/m^2 of electromagnetic radiation from the Sun, the great distance of Eris results in it being very cold. Nevertheless, its blackbody emission has been measured, using the IRAM telescope in Spain, to be 1.3 mJy (milliJanskies, where a milliJansky is 10^{-29} Watts per m^2 per Hz) at a frequency of 250 GHz.

a) Assuming Eris is rotating quickly and thus has a uniform surface temperature, and that it reflects half the light incident on it back into space, what is the radius of Eris?

b) Eris has a moon named Dysnomia, with an orbital period of 15.77 days (1.363×10^6 s) and whose orbit is consistent with being circular. At its widest point on the projection onto the sky, the Moon's orbit subtends 1.06" when viewed from 97 AU. Compute the density of Eris and compare to water. [If you weren't able to solve part a) then you may still attempt part b) assuming the radius of Eris is the same as that of Pluto: 1.165×10^6 m.]

Problem 10

In the following we will study the propagation of light in a weak gravitational field and calculate an "effective refractive index" of the gravitational field. (In this problem, we set the velocity of light in flat space, c, to 1.) For weak-field gravity we can write the gravitational metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4}$$

where $\eta_{\mu\nu}$ is the diagonal tensor with elements (-1,1,1,1) for $\mu = 0$, 1, 2 and 3 respectively and $|h_{\mu\nu}| \ll 1$. Here we use the usual harmonic coordinates, $(x^0, x^1, x^2, x^3) = (t, x, y, z)$.

a) Starting from the line element for light propagating in curved spacetime

$$g_{\mu\nu}\, dx^\mu(\lambda) dx^\nu(\lambda) = 0 \tag{5}$$

along some curve parametrized by $\lambda = t$, express the norm of the coordinate velocity, $d\vec{x}/dt$ of light in terms of $h_{\mu\nu}$ and the direction of light propagation, labelled by the unit 3-vector $\hat{k}_i$. The gravitational field is supposed to be static and $h_{it} = h_{ti} = 0$. (Repeated Greek indices are summed over 0,1,2,3.)

b) Write, using the previous result, and the assumption that $|h_{\mu\nu}| \ll 1$, the refractive index for light travelling in the direction $\hat{k}$, in the form $n(\hat{k}) = n_{ij}\hat{k}^i\hat{k}^j + \mathcal{O}(h_{ij}^2)$, finding the 3-tensor n_{ij}. (Repeated Latin indices are summed over 1,2,3. The index of refraction is defined as c/v where c is the flat space velocity of light and v is its velocity in the gravitational field.)

c) From the Schwarzschild solution, at a distance $r \equiv \sum_{i=1}^{3} x_i x_i$ from a spherical object of mass M, where r is large compared to the Schwarzschild radius:

$$h_{ij} = \delta_{ij} h_{tt} = \delta_{ij} 2GM/r. \tag{6}$$

Calculate the index of refraction, as a function of propagation direction, $\hat{k}$, for light propogating just above the surface of the sun, as would be observed, for instance, during a solar eclipse.

Constants and formulas

Coulomb force constant: $1/(4\pi\epsilon_0) = 8.988 \times 10^9$ N m^2/C^2

Proton mass: $M_P = 1.67310^{-27}$ Kg

Planck's constant: $h = 6.626 \times 10^{-34}$J s

Speed of light: $c = 2.998 \times 10^8$m/s

Boltzmann's constant: $k = 1.381 \times 10^{-23}$J/K

Fresnel reflection equation at normal incidence: The reflection amplitude for light at normal incidence, from a lossless nonmagnetic dielectric material with index of refraction n_1 to one with index of refraction n_2 is:

$$r = \frac{n_1 - n_2}{n_1 + n_2}. \tag{7}$$

Here the reflection amplitude, r, is defined so that the incident and reflected components of the electric field are:

$$\vec{E}_i(x,t) = \vec{E}_0 e^{i\omega t - ikx} \tag{8}$$
$$\vec{E}_r(x,t) = -r\vec{E}_0 e^{i\omega t + ikx} \tag{9}$$

Gravitational constant: $G = 6.673 \times 10^{-11}$m^3/(kg s^2)

Radiated power per unit area per unit frequency, ν, per unit solid angle from a black body at temperature T:

$$B(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}. \tag{10}$$

Total power per unit area (integral of B over frequency and solid angle): $P/A = \sigma T^4$ where the Stefan-Boltzman constant, σ is:

$$\sigma = \frac{2\pi^5}{15} \frac{k^4}{h^3 c^2} = 5.67 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m}^2\mathrm{K}^4}. \tag{11}$$

Astronomical unit: AU=1.496$\times$ 10^{11} m

Mass of the sun: $M_S = 1.98892 \times 10^{30}$ kg

Radius of the sun: $R_S = 6.960 \times 10^8$ m