Canadian Association of Physicists
SUPPORTING PHYSICS RESEARCH AND EDUCATION IN CANADA

UNIVERSITY OF SASKATCHEWAN

2013 CAP Prize Examination

Compiled by the Department of Physics & Engineering Physics, University of Saskatchewan

Tuesday, February 5, 2013 Duration: 3 hours.

Last name: ____________________ First name: ____________________

Institution: ________________________

Instructions:

1. You have 3 hours to complete this exam.
2. A scientific calculator is permitted but textbooks or reference materials are not.
3. There are 10 questions, each weighted equally. It is unlikely that you will be able to answer each question, therefore you should budget your time wisely.
4. Write your solutions on the pages provided. Use the back of the pages if more space is needed.
5. This exam consists of twenty three (23) pages.

Fundamental Constants:

- Atomic mass unit: $u = 1.661 \times 10^{-27}\,\text{kg} = 931.5\ \text{MeV/c}^2$
- Boltzmann's constant: $k_B = 1.381 \times 10^{-23}\ \text{J/K} = 8.617 \times 10^{-5}\ \text{eV/K}$
- Coulomb's Constant: $k = 1/4\pi\epsilon_0 = 8.988 \times 10^9\ \text{Nm}^2/\text{C}^2$
- Electron Mass: $9.109 \times 10^{-31}\,\text{kg} = 5.110 \times 10^{-1}\,\text{MeV/c}^2$
- Proton Mass: $1.673 \times 10^{-27}\ \text{kg} = 938.2\ \text{MeV}/c^2$
- Electron Compton Wavelength $\frac{h}{mc} = 2.426 \times 10^{-12}\,\text{m}$
- Electron Volt: $1\,\text{eV} = 1.602 \times 10^{-19}\,\text{J}$
- Elementary Charge: $e = 1.602 \times 10^{-19}\,\text{C}$
- Permittivity of Vacuum: $\epsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}$
- Permeability of Vacuum: $\mu_0 = 4\pi \times 10^{-7}\ \text{Hm}^{-1}$
- Planck's Constant: $\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}\ \text{J} \cdot \text{s} = 6.586 \times 10^{-16}\ \text{eV} \cdot \text{s}$
  $h = 6.626 \times 10^{-34}\ \text{J} \cdot \text{s} = 4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}$
  $hc = 1.986 \times 10^{-16}\ \text{J} \cdot \text{nm} = 1.240\ \text{keV} \cdot \text{nm}$
- Gravitational constant: $G = 6.672 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}$
- Speed of Light: $c = 2.998 \times 10^8\,\text{m/s}$
- Radius of the Sun: $R_\odot = 6.950 \times 10^5\ \text{km}$
- Minimal distance between Sun and Earth (perihelion distance): $r_p = 1.471 \times 10^8\ \text{km}$

Problem 1 : Mechanics/Gravitation

A rotating planet is an oblate spheroid whose equatorial radius a exceeds its polar radius b. The oblateness $\epsilon = 1 - b/a$ is a measure of this difference. The planet's oblateness has an effect on the gravitational field of the planet. To find the effect, one may replace the spheroid of volume V by a sphere of radius R with the same volume V. The gravitational field of the planet is then made up of two terms: one term represents the gravitational field of the sphere of radius R and the second term represents the gravitational field of a surface mass density (mass per unit area) that must be added to the gravitational field of the sphere to form the gravitational field of the actual spheroid. Find the required surface mass density in the first order of oblateness ϵ and express the final result in terms of equatorial radius a, oblateness ϵ, the uniform volume mass density of the planet ρ_0 and the latitude θ on the planet.

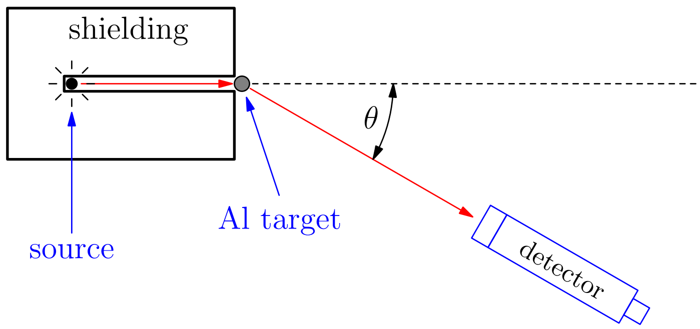
Hint: Find the distance between any point on rotating planet and the center of planet in terms of latitude.

Problem 2 : Electromagnetism

A permanently polarized sphere of radius R has a uniform polarization $\mathbf{P}$. Find the electric field inside and outside the sphere and express them in terms of $\mathbf{P}$, R, permittivity of free space and appropriate coordinates.

Problem 3 : Experimental Physics

A Compton scattering experiment is performed using a source of monoenergetic gamma rays and an aluminum target. The energies of the gamma rays scattered by the target, E', are measured for various scattering angles, θ, using equipment consisting of a scintillation crystal, photomultiplier tube, and multichannel analyzer.



(a) Derive the equation for the energy of the scattered gamma ray, E', in terms of the energy of the incident gamma ray, E, the rest energy of the electron, E_0, and the scattering angle, θ.

(b) The following data were collected:

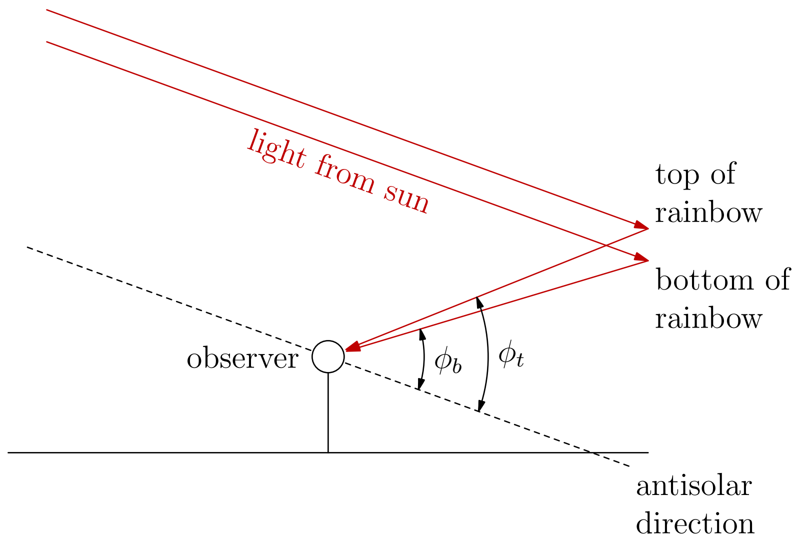
Scattering Angle, θ (degrees)	Energy of Scattered Gammas, E' (MeV)
41.4	0.500
60.0	0.402
75.5	0.336
90.0	0.288

Using these data only, calculate the rest energy of the electron and the energy of the incident gamma rays.

Problem 4 : Optics

Relative to the line from the sun and passing through an observer (the antisolar direction), calculate the range of angles at which a primary rainbow appears. The index of refraction of water for red light is 1.330 and for violet light is 1.343. You may take the index of refraction of air as 1.000.

Hint: Consider a ray of light entering the upper-half of a spherical drop of water.



Problem 5 : Quantum Mechanics

An electron on a surface experiences a two-dimensional potential

$$V(x,y) = \frac{1}{2}m\omega_1^2 x^2 + \frac{1}{2}m\omega_2^2 y^2 + Ax^4, \tag{1}$$

with $\omega_2 > \omega_1 > 0$, $A > 0$. We also assume that the ratio ω_1/ω_2 is irrational, i.e., cannot be written as a ratio of integers.

(a) Find an approximation H_0 to the Hamiltonian for which you can write down exact bound states and energy eigenvalues for this two-dimensional problem. Write down the unperturbed eigenstates and eigenvalues.
If you can't recall exact results for H_0 from your quantum mechanics course, just write in words what you remember about those eigenstates and energy levels.

(b) How much does the x^4 term in (**??**) shift the energy levels from (5a) in first order of A? In this part and the following two parts: If you do not know how to solve the problem, describe how you would attack the problem.

(c) How much does the x^4 term in (**??**) shift the eigenstates from (5a) in first order of A?

(d) Now we expose the electron to a time-dependent electric field

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E} \\ 0 \\ 0 \end{pmatrix} \exp(-|t|/\tau)$$

in x-direction. $\mathcal{E}$ and τ are constant parameters. How large is the first order transition probability between the lowest two energy levels for the Hamiltonian H_0 from (5a)?

Problem 6 : Special Relativity

A photon is emitted from a source moving in a circle at a constant speed and the same photon is received by a detector moving on the same circle at the same speed. Calculate the ratio of the detected photon energy and the emitted energy as measured in the instantaneous rest-frame of the source.

Hint: It may be useful to recall that four velocity is defined by $u = (\gamma c; \gamma \mathbf{v})$, and the four-momentum is $p = (E/c; \mathbf{p})$. From these results demonstrate that in the convention where $u \cdot u = c^2$, the energy of a photon of four-momentum p measured by an observer with four-velocity u is $E = p \cdot u$.

Problem 7 : Statistical and Thermal Physics

Choose and answer only one part: **(a)** or **(b)**

(a) Consider a gas of N noninteracting electrons in a uniform magnetic field $\mathbf{B} = B\hat{z}$ in a macroscopic system. Due to the interaction between the magnetic moment associated with the electron spin and the field, the single-particle energy ϵ is shifted by $\mu_B B$ and $-\mu_B B$ for spin "up" and "down", respectively, where μ_B is the Bohr magneton. Assume that the field affects only spin of the electron and not its orbital motion. Show that the spin magnetic susceptibility χ at arbitrary temperature is given by

$$\chi = 2\mu_B^2 \int \mathrm{d}\epsilon \, \frac{\mathrm{d}D(\epsilon)}{\mathrm{d}\epsilon} f(\epsilon) \,, \tag{2}$$

where $D(\epsilon)$ is the density of orbital states and $f(\epsilon)$ is the Fermi-Dirac distribution function,

$$f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/\tau} + 1} \,, \tag{3}$$

with fundamental temperature τ and chemical potential μ. Express χ in terms of N and τ in the thermodynamic limit ($\tau \to \infty$), assuming that the lowest single-particle energy is zero and $D(0) = 0$.

(b) Consider a gas of N free, noninteracting electrons (mass m) confined in a cubic box of side length L (volume $V = L^3$) in the ultra-relativistic limit. The relativistic single-particle energy can be approximated by $\epsilon = \sqrt{(pc)^2 + (mc^2)^2} \approx pc$. Momentum $\mathbf{p}$ is quantised as

$$\mathbf{p} = \frac{\pi\hbar}{L}(n_x, n_y, n_z) \,; \quad n_{x,y,z} = 1, 2, 3, \cdots \,. \tag{4}$$

Assume that the system is in the macroscopic limit, $L \to \infty$, and consider zero temperature. The ultra-relativistic limit cannot be reached if the electron density $n = N/V$ is too small. Express the condition $p_F c \geq mc^2$ in terms of n. Also express the internal energy U in terms of N and Fermi momentum p_F, and the pressure of the gas in terms of U and V.

Problem 8 : Particle Physics/Quantum Mechanics

Neutrino oscillations can originate from a "mismatch" between the neutrino states created by the weak interaction and the mass eigenstates. In a model with two neutrinos, the weak interaction states $|\nu_e\rangle$ and $|\nu_\tau\rangle$ are related to the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ by

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix} \tag{5}$$

where $|\nu_1\rangle$ and $|\nu_2\rangle$ respectively correspond to neutrino masses m_1 and m_2. If a weak interaction produces an electron neutrino state $|\nu_e\rangle$ of momentum $\mathbf{p}$ at $t = 0$, find an expression for the probability of detecting a tau neutrino state $|\nu_\tau\rangle$ at time t. You may assume that the neutrino masses are small compared with the momentum scale, i.e., $mc^2 \ll |\mathbf{p}|c$. Based on your results, what information on the neutrino mass spectrum could be inferred from neutrino oscillations?

Problem 9 : Condensed Matter Physics

For some metals, one can model the conduction electrons as a gas of nearly free electrons (mass m) moving in a weak periodic potential $V(\mathbf{r})$ created by the lattice of ions. One can write the single-electron wave function for a given wave number $\mathbf{k}$ as a Fourier expansion,

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{\mathbf{K}} C_{\mathbf{k}-\mathbf{K}}\, e^{i(\mathbf{k}-\mathbf{K})\cdot\mathbf{r}}\,, \tag{6}$$

where $\mathbf{K}$s are reciprocal lattice vectors.

(a) Show that $\psi_{\mathbf{k}}(\mathbf{r})$ above satisfies the Bloch theorem. Also express $V(\mathbf{r})$ in terms of Fourier transforms.

(b) By substituting the above wave function into the Schrödinger equation, obtain the Schrödinger equation in momentum space:

$$\left[E - \frac{\hbar^2}{2m}(\mathbf{k}-\mathbf{K})^2\right] C_{\mathbf{k}-\mathbf{K}} = \sum_{\mathbf{K}'} V_{\mathbf{K}'-\mathbf{K}} C_{\mathbf{k}-\mathbf{K}'}\,, \tag{7}$$

where $V_{\mathbf{K}'-\mathbf{K}}$ is the Fourier transform of $V(\mathbf{r})$ for wave vector $\mathbf{K}'-\mathbf{K}$.

(c) Consider a two-dimensional square lattice of ions with lattice spacing a. Define the first Brillouin zone by $k_x = [0, 2\pi/a]$ and $k_y = [0, 2\pi/a]$. What is the degeneracy at $\mathbf{k} = (\pi/a, \pi/a)$ in the unperturbed system?

(d) The reciprocal lattice vectors are $\mathbf{K}_{00} \equiv (0,0)$, $\mathbf{K}_{10} \equiv (1,0)$, $\mathbf{K}_{01} \equiv (0,1)$, and $\mathbf{K}_{11} \equiv (1,1)$ in units of $2\pi/a$. Thus, the possible values of $\mathbf{K}'-\mathbf{K}$ are $(0,0)$, $(\pm1,0)$, $(0,\pm1)$, and $(\pm1,\pm1)$. Explain that there are only three distinct values of $V_{\mathbf{K}'-\mathbf{K}}$ and denote them as V_{00}, V_{10}, and V_{11}. Using the notations,

$$E_{ij} = \frac{\hbar^2}{2m}(\mathbf{k}-\mathbf{K}_{ij})^2\,, \quad C_{ij} = C_{\mathbf{k}-\mathbf{K}_{ij}}\,; \quad i,j = 0,1, \tag{8}$$

and assuming $V_{(0,0)} \equiv V_{00} = 0$, rewrite Eq.(**??**) in matrix form.

(e) Find the energy eigenvalues at $\mathbf{k} = (\pi/a, \pi/a)$ for $V_{10} = 0$ and $V_{11} \neq 0$. What happens to the degeneracy at this $\mathbf{k}$-point?

Problem 10 : Applied Physics

We want to construct a device which triggers an alarm upon unauthorized entry into a house. To avoid detectability of the device by the naked eye, we want to use an ultraviolet light ray of wavelength $\lambda = 375\,\text{nm}$, corresponding to a commercially available UV laser diode. The material for the photocathode is K_2CsSb with a quantum efficiency

$$QE = \frac{\text{emitted photoelectrons}}{\text{incident photon}} = 0.30.$$

The light ray stretches across the entry door and points at a photocathode to generate a current. Termination of the current through unauthorized entry would trigger the alarm. At the location of the door that we want to secure, the most intense potential background source over all wavelength ranges is daylight. The sun has a thermal emission spectrum (spectral emittance in the wavelength scale = emitted power per area in the wavelength scale)

$$e(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp\left(\dfrac{hc}{\lambda k_B T}\right) - 1}$$

with a surface temperature $T = 5780\text{K}$.

(a) "Termination" of the photocurrent actually means dropping of the current below a threshold value I_{min}. However, continuous background sources with an ultraviolet component can generate a "dark current" which may be present if the ultraviolet light ray is interrupted. To limit this effect, we design the optics of the photocathode such that it only admits wavelengths between 374 nm and 376 nm, and we choose the diameter of the aperture of the photocathode as 1.50 mm. Furthermore, we choose the threshold current I_{min} conservatively as ten times the dark current that we expect from daylight.
How large do we have to choose the threshold current?

(b) We cannot operate the device exactly with a photocurrent at the threshold value I_{min}, because then even slightest variations in the power output of our UV light source could trigger false alarms. Therefore we would like to operate the device with a photocurrent $I = 2I_{min}$. On the other hand, as a health and safety precaution, we would like to limit the maximum power per cross section of the laser beam to less than $10^{-2}\,\text{W/cm}^2$. Assume that the laser beam has a circular cross section with diameter 1.00 mm. Can we operate the device with these specifications?