# 2014 CAP University Prize Exam

Compiled by members of the Department of Physics and Astronomy, Western University

Tuesday, February 4, 2014

Time: 3 hours

Last name:______________________________First name:________________________________

Institution:________________________________________________________________________

Instructions:

1. This exam consists of 10 questions. All have equal weight, but they are not all equally difficult.
2. The duration of the exam is three hours.
3. It is unlikely that you will be able to provide complete answers to all 10 questions in the time provided. Budget your time accordingly.
4. This exam has 23 pages. Write your name and institution in the space provided *on every page.*
5. Write your answers on the pages provided. You may use the back of the pages if required.
6. A scientific calculator is permitted. No textbooks or other references are allowed.
7. No computers, cell phones, MP3 players, or devices capable of connecting to the internet are permitted.
8. Have fun!

Some constants:

$c = 3.00\times10^{8}$ m s$^{-1}$

$e = 1.60\times10^{-19}$ C

$\hbar = 1.05\times10^{-34}$ J s

$\varepsilon_0 = 8.85\times10^{-12}$ C$^2$/(N m$^2$)

$m_e = 9.11\times10^{-31}$ kg

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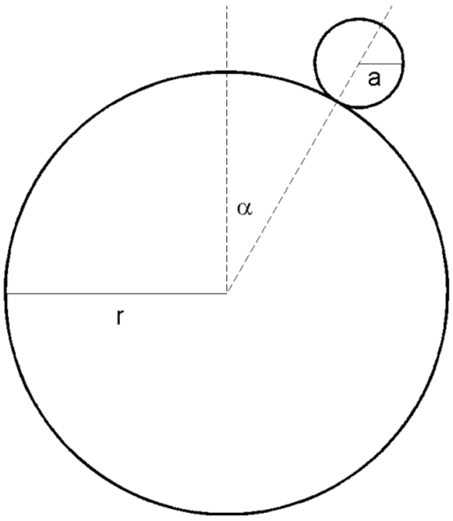
1. A ball of radius $a$ rolls without slipping down the surface of a fixed rough sphere of radius $r$, starting from rest at the top. The angle $\alpha$ is defined as shown in the figure. Show that the ball leaves the surface of the sphere when $\cos\alpha = 10/17$ .

*Hint: The moment of inertia of a sphere of mass m and radius R is* $\frac{2}{5}mR^2$ .



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2. The motion of a medium such as water influences the speed of light passing through it. This effect was first observed by Fizeau in 1851.

   Consider a beam of light directed along the axis of a horizontal pipe filled with water flowing with a velocity $v$. Determine the speed $u$ of the light measured in the lab frame when the beam travels in the same direction as the flow of the water. Find an approximation to this expression valid when $v$ is much smaller than $c$.

*Hint: The Lorentz law for the addition of velocities is* $u = \dfrac{u' + v}{1 + \dfrac{u'v}{c^2}}$.

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3. Picture the electron as a uniformly charged shell with total charge $e$ and radius $R$. The potential energy stored in an electric field is given by

$$U_e = \frac{1}{2}\int \varepsilon_0 E^2 dV ,$$

where the integral is over all space.

a. Calculate the total potential energy $U_e$ contained in the electric field.

b. Assume that all of the electron mass comes from the potential energy $U_e$ of the electric field, so that

$$U_e = m_e c^2 .$$

Determine the radius of the electron in this scenario.

*Hint: In spherical polar coordinates,* $dV = r^2 \sin\theta dr d\theta d\phi$. $\phi$ *runs from 0 to* $2\pi$ *and* $\theta$ *from 0 to* $\pi$.

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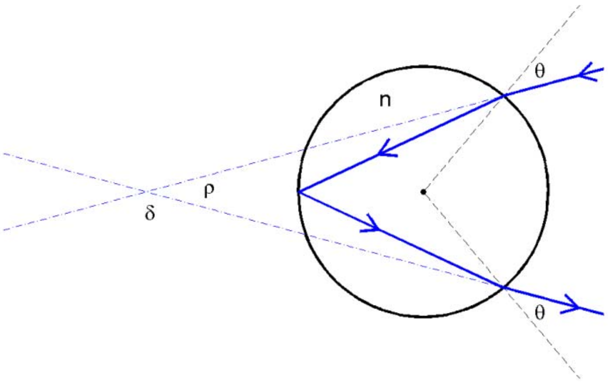
4. A rainbow is formed by light rays which pass through raindrops as shown in the figure at right. The dot indicates the center of the raindrop. The solid line is the path followed by a ray. The angle of incidence of the ray is $\theta$. $\delta$ is the angle by which the path of the ray is deviated from its original direction. The angle $\rho$ is the angular radius of the rainbow and is equal to $\pi - \delta$. Given that the actual value of $\rho$ is that for which $\frac{d\rho}{d\theta} = 0$, calculate $\rho$ in terms of the refractive index of the raindrop, $n$. Take the refractive index of air to be 1.



Calculate the numerical values of $\rho$ for blue and red light, given that $n = 1.3439$ for blue and 1.3316 for red, and use your results to explain the order of colours in a real rainbow.

*Hint:* $\frac{d}{d\theta}\arcsin u = \frac{1}{\sqrt{1-u^2}}\frac{du}{d\theta}$

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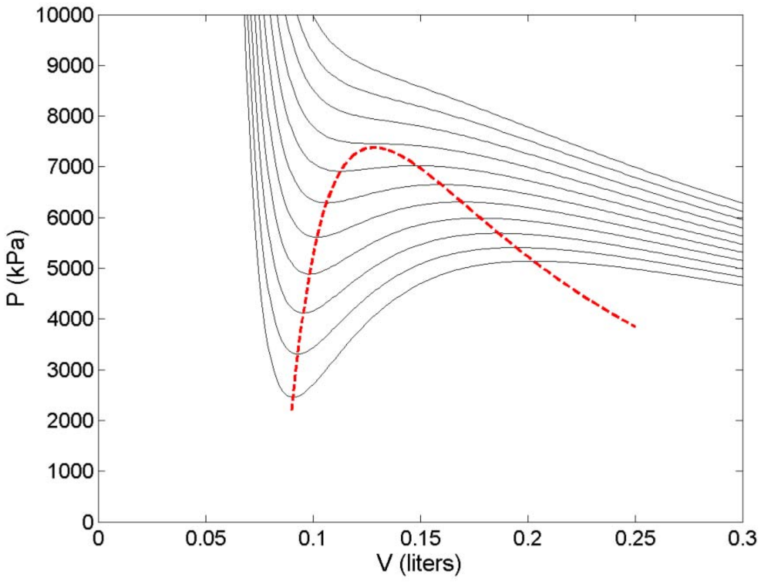
5.

5. The van der Waals equation of state is

$$\left(P+\frac{a}{V^2}\right)(V-b)=RT$$

where $P$ is the pressure, $V$ is the volume of the container, and $T$ is the absolute temperature. $a$ and $b$ are constants that depend on the material. For $CO_2$, $a$ = 365.8 kPa $L^2/mol^2$ and $b$ = 0.04286 L/mol (kPa = kilopascals; L = liters; mol = moles). The figure shows the pressure-volume phase diagram for $CO_2$ calculated using this equation of state. The solid black curves are isotherms, that is, each black curve shows pressure as a function of volume for a fixed value of $T$. The dashed curve is called the spinodal curve. It joins the turning points of the isotherms, i.e., it is the locus of points at which $\left(dP/dV\right)_T = 0$.



a) Draw an arrow on the figure pointing from low to high temperatures at constant $V$.

b) The region of the phase diagram inside the spinodal curve is thermodynamically unstable. Explain why. Explain what happens to the system if it is initially prepared at a pressure, volume and temperature inside this unstable region. The regions to the left and right of the spinodal curve are thermodynamically stable. What do these regions correspond to physically?

c) Find the equation for $P(V)$ on the spinodal curve in terms of $a$ and $b$.

d) The peak of the spinodal curve is called the critical point of the material. Find the pressure, volume, and temperature at the critical point in terms of $a$ and $b$. Show that for $CO_2$, $P_c = 7370$ kPa, $V_c = 0.129$ L, and $T_c = 304$ K.

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6.

a. Estimate the depth of the deepest hole that could be drilled into the Earth's crust. Assume that the crust is made of granite, which has strength 300 MPa and density 2700 kg/m$^3$. Comment on the anticipated validity of your estimate.

b. Design (crudely) a device to drill such a hole, and, in the process, record the temperature as a function of depth and return rock samples to the surface. What factors have to be taken into account in your design?

c. Discuss why it would be scientifically interesting to drill such a hole.

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7. The material in a star is pulled towards the center of the star by gravity. This inward force is balanced by an outward force due to a pressure gradient within the star, i.e., the pressure increases as you move further into the star. Consider the star as being made up of a series of spherical shells of thickness $dr$. Each shell has a radius $r$, density $\rho(r)$, and mass $dM(r)$.

   a) Find an expression for $dM/dr$.

   b) By considering the forces acting on a single shell, find an expression for the pressure gradient $dP/dr$, where $P$ is the pressure.

   c) Calculate the pressure $P(r)$ inside a star of total mass $M_S$, radius $R_S$, and central density $\rho_c$ if the density varies with $r$ as

$$\rho(r) = \rho_c\left(1 - \frac{r}{R_S}\right).$$

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8. The velocity field $\bar{v} = (u, v, w)$ in a fluid is described by the Navier-Stokes equation,

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} = -\frac{\nabla p}{\rho} + \bar{g} + \eta \nabla^2 \bar{v} .$$

For an incompressible fluid, we also have

$$\nabla \cdot \bar{v} = 0 .$$

Here $t$ is the time, $\rho$ the fluid density, $\eta$ the kinematic viscosity, $p$ the pressure, and $g$ the acceleration due to gravity.

Consider an incompressible viscous fluid that occupies the space $0 < y < \infty$ above an infinite flat plate in the $y = 0$ plane. The plate oscillates in the $x$ direction with velocity $U\cos(\omega t)$. Assume that the velocity of the fluid goes to zero as $y \to \infty$. Also assume that the fluid does not slip on the surface of the plate, so the velocity of the fluid at $y = 0$ is the same as the velocity of the plate. No external pressure gradients are applied and we will neglect gravity, so $\nabla p = \bar{g} = 0$.

a) Explain qualitatively what each component of the velocity field in the fluid will look like as a function of position and time, and why.

b) Simplify the Navier-Stokes equation appropriately and solve to find the velocity field.

c) Sketch your solution for the $x$-component of the velocity field as a function of $y$ for a particular value of $t$.

*Hint:* $i^{1/2} = \frac{1}{2^{1/2}}(1+i)$.

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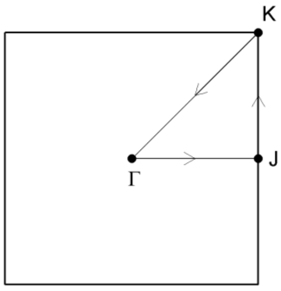
9. Consider an insulator with a simple cubic lattice structure and lattice constant *a*. The *x*-*y* plane is a (001) crystal plane. The energy of lowest conduction band of the insulator is given by the dispersion relation

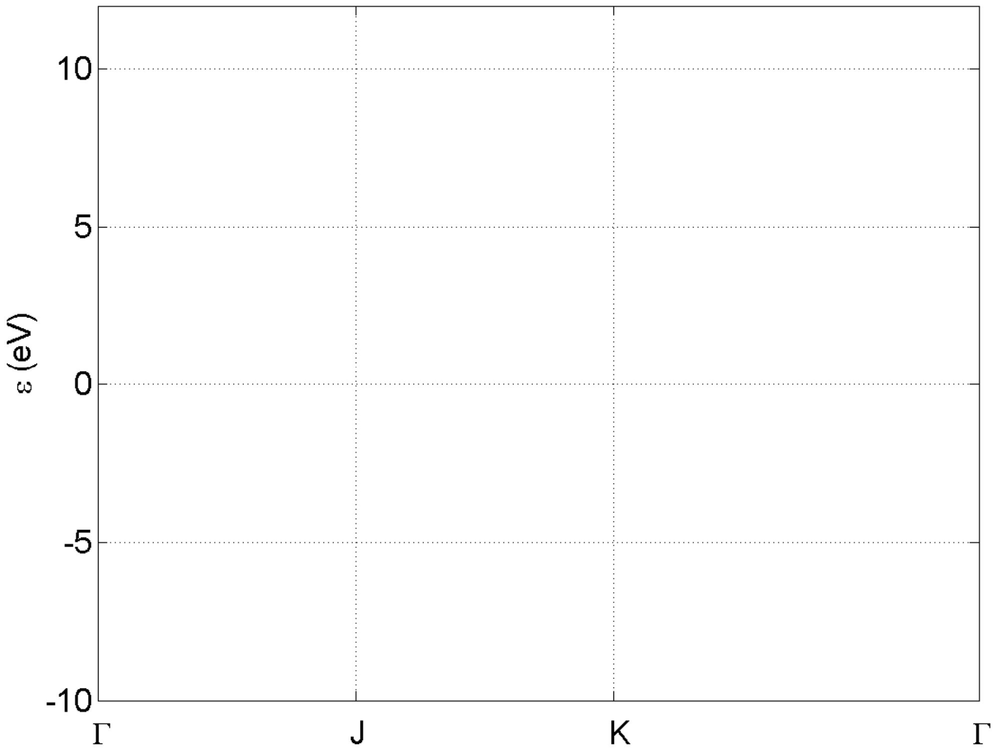
$$\varepsilon(k_x, k_y, k_z) = E_0 + 2V_0[\cos(k_x a) + \cos(k_y a) + \cos(k_z a)],$$

where $E_0 = 3.2$ eV and $V_0 = 1.1$ eV. Here $\bar{k} = (k_x, k_y, k_z)$ is a vector in reciprocal space. Certain high-symmetry points in the $k_x - k_y$ plane are given special names, as indicated in the diagram at right. The point $\Gamma$ is at $(k_x, k_y) = (0,0)$; J is at $(k_x, k_y) = (0, \pi/a)$; and K is at $(k_x, k_y) = (\pi/a, \pi/a)$.



a) Plot the minimum and maximum energies of the above conduction band along the line segments $\Gamma - \mathrm{J}$, $\mathrm{J} - \mathrm{K}$, and $\mathrm{K} - \Gamma$ in the $k_x - k_y$ plane, taking $\cos k_z a$ to range from $-1$ to +1. Draw your plot using the template provided below.

b) The Fermi energy of this system is –3.32 eV. Show this energy on your plot and indicate which regions of the conduction band are occupied and which are empty.



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10. The “Old Quantum Theory” was based on a heuristic prescription known as the Bohr-Sommerfeld quantization rule which was used to determine which states of a classical system were allowed quantum states. In one dimension, the Bohr-Sommerfeld quantization rule is

$$\oint p\,dx = nh$$

where $p$ is the momentum, $x$ the corresponding coordinate, $h$ Planck’s constant, and $n$ a quantum number. The integral is over a closed path. Use this rule to calculate the allowed energy levels of a ball which is bouncing elastically in the vertical direction.

*Hint:* $\int (ax+b)^{1/2}dx = \dfrac{2(ax+b)^{3/2}}{3a}$.

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