Canadian Association of Physicists
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UNIVERSITY OF MANITOBA

2015 Lloyd G. Elliott University Prize Exam

Compiled by the Department of Physics & Astronomy, University of Manitoba

Tuesday, February 3, 2015 Duration: 3 hours.

Last name: ____________________ First name: ____________________

Institution: ______________________________

Instructions:

1. You have 3 hours to complete this exam.
2. A scientific calculator is permitted but textbooks or reference materials are not.
3. There are 10 questions, each weighted equally. It is unlikely that you will be able to answer each question, therefore you should budget your time wisely.
4. Write your solutions on the pages provided. Use the back of the pages if more space is needed.

Fundamental Constants:

- Atomic mass unit: $u = 1.661 \times 10^{-27}\,\text{kg} = 931.5\ \text{MeV/c}^2$
- Boltzmann's constant: $k_B = 1.381 \times 10^{-23}\ \text{J/K} = 8.617 \times 10^{-5}\ \text{eV/K}$
- Coulomb's Constant: $k = 1/4\pi\epsilon_0 = 8.988 \times 10^9\ \text{Nm}^2/\text{C}^2$
- Electron Mass: $9.109 \times 10^{-31}\,\text{kg} = 5.110 \times 10^{-1}\,\text{MeV/c}^2$
- Proton Mass: $1.673 \times 10^{-27}\ \text{kg} = 938.2\ \text{MeV}/c^2$
- Electron Compton Wavelength $\frac{h}{mc} = 2.426 \times 10^{-12}\,\text{m}$
- Electron Volt: $1\,\text{eV} = 1.602 \times 10^{-19}\,\text{J}$
- Elementary Charge: $e = 1.602 \times 10^{-19}\,\text{C}$
- Permittivity of Vacuum: $\epsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}$
- Permeability of Vacuum: $\mu_0 = 4\pi \times 10^{-7}\ \text{Hm}^{-1}$
- Planck's Constant: $\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}\ \text{J}\cdot\text{s} = 6.586 \times 10^{-16}\ \text{eV}\cdot\text{s}$
  $h = 6.626 \times 10^{-34}\ \text{J}\cdot\text{s} = 4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}$
  $hc = 1.986 \times 10^{-16}\ \text{J}\cdot\text{nm} = 1.240\ \text{keV}\cdot\text{nm}$
- Gravitational constant: $G = 6.672 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}$
- Speed of Light: $c = 2.998 \times 10^8\,\text{m/s}$
- Radius of the Sun: $R_\odot = 6.950 \times 10^5\ \text{km}$
- Minimal distance between Sun and Earth (perihelion distance): $r_p = 1.471 \times 10^8\ \text{km}$

Problem 1 : Classical Mechanics

A yo-yo consists of two disks of mass M and radius R connected by a shaft of mass m and radius r; a weightless string is wrapped around the shaft. The two disks and rod have uniform mass distribution. The moment of inertia of each disk is $I_d = \frac{1}{2}MR^2$ while that of the shaft is $I_s = \frac{1}{2}mr^2$.

(a) The free end of the string is held stationary in Earth's gravitational field. Assuming that the string starts out vertical, find the motion of the yo-yo's centre of mass.

(b) The yo-yo is transported to empty space, where there is no gravitational field, and a force F is applied to the end of the string. Describe the motion of the centre of mass of the yo-yo, the yo-yo rotation and the motion of the free end of the string.

Problem 2 : Quantum Mechanics (Question 1)

Consider a 2-dimensional vector space spanned by the basis vectors:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Suppose a system is described by the Hamiltonian:

$$H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$$

where g and h are real, positive constants.

(a) Solve the eigenvalue problem for H to find the energy eigenstates and the eigenenergies ϵ_+ and ϵ_-, where $\epsilon_+ > \epsilon_-$.

(b) Given that the system starts out in state $|1\rangle$ at time $t = 0$, show that the system, at time t, is described by the state vector:

$$|\Psi(t)\rangle = e^{-iht/\hbar} \begin{pmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{pmatrix}$$

(c) What is the probability that a measurement of the energy of the system in state $|\Psi(t)\rangle$ at time $t = t_0$ yields the value ϵ_-? Having measured ϵ_- at $t = t_0$, the measurement is repeated at time $t = t_0 + \Delta t$. What is the probability that the other eigenenergy, ϵ_+, is now measured?

Problem 3 : Quantum Mechanics (Question 2)

A hydrogen atom is placed in a time dependent electric field given by

$$\mathbf{E}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \hat{z}\, E_0 e^{-\gamma t} & \text{if } t > 0 \end{cases}$$

What is the probability (to first order) that, as $t \to \infty$, the atom has made a transition from the ground state to the $2p$ state? The normalized wavefunctions $\Psi_{n,l,m}$ are given by

$$\begin{aligned} \Psi_{1,0,0}(r,\theta,\phi) &= \frac{1}{\sqrt{\pi\, a_o^3}}\, e^{-r/a_o} \\ \Psi_{2,1,1}(r,\theta,\phi) &= -\frac{1}{8\sqrt{\pi\, a_o^5}}\, r\, e^{-r/2a_o}\, \sin\theta\, e^{i\phi} \\ \Psi_{2,1,0}(r,\theta,\phi) &= \frac{1}{\sqrt{32\pi\, a_o^5}}\, r\, e^{-r/2a_o}\, \cos\theta \\ \Psi_{2,1,-1}(r,\theta,\phi) &= \frac{1}{8\sqrt{\pi\, a_o^5}}\, r\, e^{-r/2a_o}\, \sin\theta\, e^{-i\phi} \end{aligned}$$

where a_o is the Bohr radius.

Problem 4 : Statistical Mechanics

Bosons in one dimension: Consider bosons on a lattice in one dimension with dispersion relation $\varepsilon_k = \Delta + \frac{k^2}{2m}$. For this problem we set $\hbar = k_B = 1$. Δ is an excitation gap, and m the mass of the bosons. The allowed momenta are $k = \frac{2\pi}{Ma}n$, $n = 0, \cdots, M-1$ where M is the number of sites in the lattice, and a is the lattice constant.

(a) Derive an expression for the grand canonical potential per site:

$$\Omega = -\frac{T}{M}\ln Z_g \qquad \text{with} \qquad Z_g = \mathrm{Tr}\,\exp[-(\hat{H} - \mu\hat{N})/T]\,,$$

at temperature T for the special case of zero chemical potential, $\mu = 0$. Here $\hat{H} = \sum_k \varepsilon_k \hat{n}_k$ where $\hat{n}_k$ is the occupation number operator, and $\hat{N} = \sum_k \hat{n}_k$.

(b) Simplify the expression for the grand canonical potential Ω if $M \to \infty$ and $T \ll \Delta$. Based on this simplified expression show that the specific heat, $c(T) = -T\partial^2\Omega/\partial T^2$, at low temperatures T scales as $c(T) \sim T^{-3/2}\exp(-\Delta/T)$.

Problem 5 : Solid state physics

Germanium assumes a crystal structure with two atoms per unit cell, and so the phonon dispersion relations have three acoustical and three optical branches.

(a) Sketch the phonon dispersion relations for waves propagating in a typical direction. Be sure to indicate the qualitative behaviour at $k = 0$ and at the boundary of the first Brillouin zone. Indicate the optical and acoustical branches.

(b) Explain physically why the acoustical modes contribute more to the low temperature specific heat of this solid than the optical ones.

(c) The phonon modes can be considered independent harmonic oscillators, each with frequency $\omega_\lambda(\mathbf{k})$. From basic statistical mechanics, show that the thermal equilibrium occupancy of phonons of mode λ and wavevector $\mathbf{k}$ is

$$\langle n_\lambda(\mathbf{k})\rangle = \frac{1}{e^{\beta\omega_\lambda(\mathbf{k})} - 1} \tag{1}$$

(d) In the Debye model, the dispersion relations are approximated by a simple linear function: $\omega(\mathbf{k}) = vk$. In the Einstein model, they are approximated by a constant: $\omega(\mathbf{k}) = \omega_E$. Describe how, and why, a combination of these models can be used to approximate the dispersion relations of Ge. What limits are there on k and ω in these models?

(e) Use the model of part (d) to calculate the following expression for the lattice specific heat for Ge at low temperatures.

$$C_V = 3Nk_B\left[\frac{4\pi^4}{5}\left(\frac{T}{\Theta_D}\right)^3 + \left(\frac{\hbar\omega_E}{k_B T}\right)^2 e^{-\hbar\omega_E/k_B T}\right] \tag{2}$$

where N is the number of unit cells in the sample, and $\Theta_D = \hbar\omega_D/k_B$ is known as the Debye temperature, with ω_D the maximum acoustical mode frequency in this model.

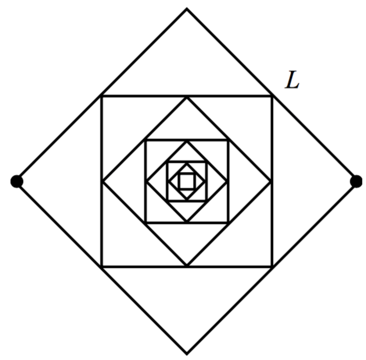
Sketch your result, and relate the qualitative features of your graph to your answer in part (b). What does “low temperature” mean in this context?

Hint: You will need

$$\int_0^\infty \frac{x^4 e^x}{(e^x - 1)^2}\,dx = \frac{4}{15}\pi^4 \tag{3}$$

Problem 6 : Electromagnetism (Question 1)

The electric circuit in the figure is made from an infinite number of squares of identical wire. The largest square has a side of length L, and each side has resistance R. Each subsequent smaller square connects the centres of each side of the larger square. What is the electrical resistance between opposite corners of the largest square?



Problem 7 : Electromagnetism (Question 2)

A long solid dielectric cylinder with circular cross-section of radius R is permanently polarized so that the polarization vector **P** everywhere within the dielectric points radially outward from the axis of the cylinder with a magnitude which is proportional to the distance from the axis, that is, $\mathbf{P} = P_0 \boldsymbol{\rho}$ where P_0 is a positive constant and $\boldsymbol{\rho}$ is the radial vector in a cylindrical coordinate system. Show that the total work expended by an external agent per unit length of the cylinder to rotate the cylinder around its axis from rest to a final angular velocity ω is $W = (1/6)\pi\mu_0 P_0^2 \omega^2 R^6$. Neglect end effects, assume that the rotation has no affect on the polarization **P**, and treat the problem in the non-relativistic limit $\omega R \ll c$. Hint: consider the energy stored in the electromagnetic field.

Problem 8 : Subatomic Physics

The Philae lander arrived on the surface of comet 67P/Churyumov Gerasimenko on November 12, 2014. Unfortunately, it ended up with its solar panels largely obstructed by a cliff. As a more robust – but harder to come by – alternative, the designers could have chosen a radioisotope thermoelectric generator (RTG). In an RTG, a pellet containing ^{238}Pu is creating heat via alpha decay, which is converted into electricity with thermocouples.

If the Philae lander needs 32 W of electric power, how many kilograms of ^{238}Pu had to be loaded into the RTG at lift-off (March 2nd, 2004)? The half-life of ^{238}Pu is 87.7 years, the energy release in a single alpha decay is 5.593 MeV, Avogadro's number is 6.022×10^{23}, and the efficiency of the thermocouples is 5 %.

Problem 9 : Astrophysics

Gravitational Stability of Self-Gravitating Cylinders: Astrophysicists have long studied the equilibrium and stability of spherical gas clouds that are supported against self-gravity by their internal pressure. This problem develops the framework for a theory of self-gravitating cylinders, which are common in star-forming regions. The problem can be solved using only the equation of hydrostatic equilibrium, the equation of state for an isothermal gas, and Newton's theory of gravity.

Consider an infinitely long cylinder with mass per unit length m, containing a radial density distribution $\rho(r)$, which is in equilibrium between gravitational and pressure forces, such that

$$\frac{dP}{dr} = -\rho(r)g(r),$$

where $g(r)$ is the magnitude of the local gravitational acceleration at radius r. The gas is isothermal, such that the gas pressure is given by $P(r) = \sigma^2\rho(r)$, where σ is a constant. The cylinder is bounded at a finite radius r_S, where the pressure $P(r_S)$ becomes equal to the constant pressure P_S of the surrounding gas.

(a) Poisson's equation for gravity is given by $\nabla^2\Phi = 4\pi G\rho$, where Φ is the gravitational potential, and G is the gravitational constant. Use Poisson's equation to derive a formula for $g(r)$.

(b) Without assuming any particular radial density distribution, prove that a cylinder in equilibrium must exactly obey the equation

$$2(\langle P\rangle - P_S)A_S = m^2 G,$$

where $A(r) = \pi r^2$ is the cross-sectional area inside radius r, $A_S = A(r_S)$, and $\langle P\rangle$ is the volume average pressure within the cylinder, defined by $\langle P\rangle = A_S^{-1}\int_{A=0}^{A_S} P(r)dA$. (Note that averaging over cross-sectional area is the same as averaging over volume since the cylinder is infinite in length.)

(c) Demonstrate that there is a finite upper limit m_{max} to the mass per unit length, such that $m \le m_{max}$ for all isothermal cylinders that are in equilibrium. Derive a formula for m_{max}.

(d) Assume that the mass per unit length is given by $m = \alpha m_{max}$, where $0 < \alpha < 1$. Show that the cylinder obeys the equation

$$P_S = \alpha(1-\alpha)\frac{m_{max}^2 G}{2A_S}.$$

(e) Can such a cylinder be made to undergo gravitational collapse if it is squeezed by increasing P_S? Explain your reasoning.

Problem 10 : Medical/Nuclear Physics

A sphere of water with a mass of 20g contains 3700 MBq of I-131.

(a) Use the data provided below to estimate, as accurately as possible, the absorbed dose in joules per kilogram (Si unit of absorbed dose = gray or Gy) to the sphere, clearly stating the approximations you have used.

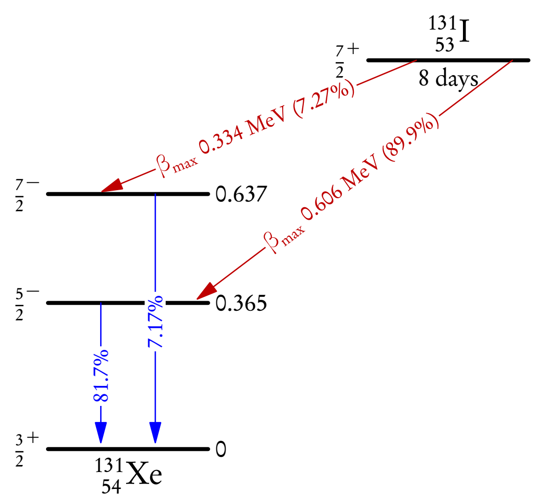
(b) Discuss the expected accuracy of your result.

Notes:
1 becquerel (Bq) (SI unit of Activity) = 1 disintegration per second
1 electron volt (eV) = 1.60218×10^{-19} joules
Density of Water = 1 g/cm^3
The CSDA (continuous-slowing-down approximation) Range is given by:

$$R_{\text{CSDA}}(E) = \int_0^E \frac{dE'}{S(E')}$$

where E is the energy and S is the Stopping Power.

Electron Kinetic Energy (MeV)	Collision Stopping Power in Water (MeV cm^2/g)	R_{CSDA} (g/cm^2)
3.00E-01	2.36E+00	8.42E-02
3.50E-01	2.24E+00	1.06E-01
4.00E-01	2.15E+00	1.29E-01
4.50E-01	2.08E+00	1.52E-01
5.00E-01	2.03E+00	1.77E-01
5.50E-01	2.00E+00	2.01E-01
6.00E-01	1.96E+00	2.27E-01
7.00E-01	1.92E+00	2.78E-01



Photon Energy (MeV)	Mass Attenuation Coefficient in water (cm^2/g)	Mass Absorption Coefficient in water (cm^2/g)
3.64E-01	1.10E-01	3.25E-2
6.37E-01	8.72E-02	3.27E-2