2017 Lloyd G. Elliott University Prize Exam

Compiled by the Department of Physics, University of Toronto

Tuesday, 7 February, 2017 Duration: 3 hours.

Last name: ______________________ First name: ______________________

Institution: ______________________________________

Instructions:

1. You have 3 hours to complete the exam.
2. A non-programmable calculator is permitted but textbooks or reference materials are not.
3. There are 10 questions, not all of equal difficulty, but each worth the same number of marks. It is unlikely that you will be able to answer each question, therefore you should budget your time wisely.
4. Write your solutions on the pages provided. Use the back of the pages if more space is needed.

Fundamental constants:

$e = 1.60 \times 10^{-19}$ C

$m_e = 9.11 \times 10^{-31}$ kg

$h = 6.63 \times 10^{-34}$ Js

$\hbar = 1.06 \times 10^{-34}$ Js

$a_\circ = 0.529 \times 10^{-10}$ m

1 Ry $= 13.606$ eV

$c = 3.0 \times 10^{8}$ m/s

$N_A = 6.02 \times 10^{23}$

$k_B = 1.38 \times 10^{-23}$ J/K

$R = 8.314$ J/K·mol

$\mu_B = 9.274 \times 10^{-24}$ J/T

$\mu_N = 5.05 \times 10^{-27}$ J/T

$m_p = 1.673 \times 10^{-27}$ kg

$\mu_\circ = 1.257 \times 10^{-6}$ H/m

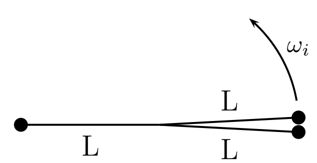
$\epsilon_\circ = 8.854 \times 10^{-12}$ F/m

1. Three identical point masses M are joined to a common central point P by massless strings of inextensible length L. Initially, one of the masses is held fixed and the other two masses are rotated around it with angular velocity ω_i, as shown in the "before" figure below. The fixed mass is then released. You may neglect gravity in this problem and assume all motion takes place in the plane.

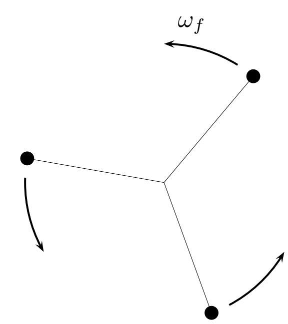
   Assume that because the centre of mass and the axis of rotation are initially different the system after release will be *unstable* and will quickly adjust to the symmetric configuration shown in the "after" figure below, which shows the motion in the centre of mass coordinates of the system.

   (a) Calculate the system's centre of mass velocity, and the system's angular velocity ω_f about its centre of mass, after adjustment. (The initial angle between the two masses on the right in the "before" figure may be neglected.)

   (b) Suppose that the system is aimed at a target T located at a distance much greater than L: what would be the best strategy for hitting T? In your answer, use the strategy of aiming the CoM of the system at T. If the system is released when the strings are horizontal (as pictured), describe the subsequent motion of the centre of mass.



before



after (CofM frame)

2. An airplane is flying in a straight line at a constant height h above the ground. The speed of the airplane, v, is much less than the speed of sound, $v_s = 343$ m/s.

   (a) Treating the engine sound as though it has a single well-defined frequency $f_\circ$ (the frequency in the rest frame of the airplane), show that an observer on the ground, directly under the path of the airplane, who measures the period T_{obs} of the arriving sound waves, will obtain a frequency

$$f_{obs} = \frac{1}{T_{obs}} = \frac{f_\circ}{1 + \frac{\sqrt{(x+v/f_\circ)^2+h^2}}{v_s/f_\circ} - \frac{\sqrt{(x^2+h^2)}}{v_s/f_\circ}}$$

   where the x is the horizontal distance between the observer and the airplane when the sound is emitted.

   (b) Show that in the limit $x \gg h$, f_{obs} reduces to

$$f_{obs} = \frac{f_\circ}{1 \pm v/v_s}.$$

   (c) Sketch a plot of $f(x)$ vs. x from part (a), assuming that $f = 200.0$ Hz, $v = 50.0$ m/s and $h = 500.0$ m.

   (d) If you could accurately measure frequency vs. time, explain how you could tell how high overhead an airplane has passed (i.e. determine h)?

3. An ideal compressor can be visualized as working in three steps: (i) starting with zero volume, an intake valve is opened to the atmosphere and then a piston is withdrawn so that air enters the compressor at 300 K and a fixed pressure of 1 atmosphere; (ii) the intake valve is closed and then the piston compresses the gas adiabatically to 2 atmospheres, and (iii) an outlet valve is opened and then the piston continues to advance, so that all of the compressed air, at a fixed pressure of 2 atm, is delivered to a pipe. Assume that all three steps are carried out reversibly (this being an ideal compressor).

   (a) Draw the p, V diagram of this process, and use this diagram to explain why the net work done by the compressor is equal to $\int V\, dp$.

   (b) Show that this work is equal to the change in the enthalpy of the gas between steps (i) and (iii), and carry out the integral to show that

   $$\text{Net work by compressor} = \frac{-\gamma}{1-\gamma}\left(p_f V_f - p_i V_i\right),$$

   where γ is the rato of the specific heat at constant pressure to that at constant volume.

   (c) If the compressor consumes 200 W of useful power, what volume of air will it deliver per second? [Assume γ=1.4. 1 atm of pressure is approximately 1.0×10^5 Pa.]

4. Positronium is a hydrogen-like system consisting of an electron and a positron bound together by the Coulomb interaction. Electrons and positrons have identical mass and spin ($\frac{1}{2}\hbar$). Their charges are equal in magnitude, but opposite in sign.

   (a) The binding energy of a hydrogen atom depends only on (i) the reduced mass of the electron-proton system, (ii) the speed of light, and (iii) the dimensionless fine structure constant, α ($\approx$ 1/137). Using dimensional analysis (or your alternative favourite method), estimate how the energy of the $1S$ – $2P$ transition in hydrogen scales with the mass of the electron.

   (b) Given that a hydrogen atom in its $2P$ state radiates a photon with a wavelength of 121.5 nm when it decays to the $1S$ state, what is the wavelength of the light emitted by a positronium atom when it decays from its $2P$ state?

   (c) Just as in hydrogen, the energy eigenstates of the positronium atom can be labeled by the orbital angular momentum ℓ of the electron and positron, and their total spin angular momentum S. What values can the total spin S of the electron and positron take?

   (d) Electrons and positrons can be modeled as spinning charge distributions, and they have permanent magnetic moments due to their spin. The magnetic moment of an electron is $\approx$ -1 μ_B. What is the magnetic moment of a positronium atom due to the spins of the electron and positron?

5. A top quark, with a mass $m_{top} = 172.5\ \mathrm{GeV/c^2}$, is produced in a proton-proton collision. It decays into a bottom quark and a W intermediate vector boson.

   (a) If the top quark is produced at rest, what is the energy of the bottom quark? Assume that the bottom quark is massless and the W boson has a mass $m_W = 80\ \mathrm{GeV/c^2}$.

   (b) If the top quark is produced with a momentum of 100 $\mathrm{GeV/c}$, what is the minimum momentum of the bottom quark?

   (c) The bottom quark in (b) decays a picosecond after being produced, as observed in its rest frame. The mass of the bottom quark is 5 $\mathrm{GeV/c^2}$ (small enough that we could ignore it in the answers to (a) and (b)). What is the minimum distance the bottom quark travels in the laboratory frame? What is the maximum distance?

6. One way of representing simple harmonic oscillation is using the complex ordinary differential equation

$$\frac{dZ}{dt} = i\omega Z, \tag{1}$$

which has solution $Z(t) = Z(0)e^{i\omega t}$. Suppose we want to test different computer time stepping algorithms by applying a finite difference approximation to this system.

(a) The *Euler forward* method replaces the ordinary derivative on the LHS of (1) with a finite difference approximation as follows:

$$\frac{dZ}{dt} \approx \frac{Z(t+\delta t) - Z(t)}{\delta t}.$$

The RHS is evaluated at time t, resulting in

$$\frac{Z(t+\delta t) - Z(t)}{\delta t} \approx i\omega Z(t). \tag{2}$$

This is a pretty good approximation as long as $\delta t\,\omega \ll 1$. Given $Z(t = 0)$, develop an algorithm to calculate $Z(n\delta t)$ using this approximation. Can you identify a problem with applying Euler forward to this system?

(b) In the *implicit* method, we replace (2) with the following:

$$\frac{Z(t+\delta t) - Z(t)}{\delta t} \approx i\omega Z(t+\delta t). \tag{3}$$

Calculate $Z(n\delta t)$ given $Z(t = 0)$ for the implicit method. Do you see a potential problem with applying the implicit method to this system?

(c) Can you come up with a method along similar lines of those in a) and b) but does not suffer from the problems you identified?

7. The Navier-Stokes equation, neglecting gravitational or other body forces, is given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u}$$

where $\boldsymbol{u}$ is the fluid velocity, p is the fluid pressure, ρ is the density and ν is the kinematic viscosity. The mass continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

A fluid of constant density and viscosity is located in the space between infinite concentric cylinders that are coaxial with the z axis. The inner cylinder has radius r_1 and is moving at a steady velocity U_z in the $\hat{z}$ direction. The outer cylinder has radius $r_2 > r_1$ and is at rest.

(a) Determine the velocity $\boldsymbol{u}(r, \phi, z)$ for the fluid at steady state, where in cylindrical coordinates (r, ϕ, z) are the radial, azimuthal and vertical coordinates. In addition, the fluid velocity is $\boldsymbol{u} = (u_r, u_\phi, u_z)$ and the gradient, Laplacian, and divergence are

$$\begin{aligned} \nabla &= \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \phi}, \frac{\partial}{\partial z} \right), \\ \nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \qquad \text{and} \\ \nabla \cdot \boldsymbol{A} &= \frac{1}{r}\frac{\partial (r A_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \end{aligned}$$

where in the last equation $\boldsymbol{A} = (A_r, A_\phi, A_z)$ is any vector field and its cylindrical coordinate components. Assume no external pressure is applied, that all fields are indepenent of ϕ, and that $\boldsymbol{u}$ is independent of z. State any other assumptions you make.

(b) The *shear stress* normal to the surface of a cylinder is the viscous force per unit area. In cylindrical coordinates, it is given by $\tau_{zr} = 2\rho\nu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right)$. What is the force per unit length along the z axis required to maintain the motion of the inner cylinder?

8. X-ray free-electron lasers (XFELs) are capable of producing 40 femtosecond pulses of x-rays, tunable between a few hundred eV and more than 10 keV in photon energy. Typical beam parameters are a beam diameter of 400 μm and 3 mJ of photons per pulse.

   (a) Star Trek IV – The Voyage Home contains a scene in which Chief Engineer Scotty proposes to make a whale tank out of 'transparent aluminum'. XFELs are capable of making transparent aluminum, very briefly, by saturable absorption: promoting virtually all the innermost bound electrons to unbound states.

   For a solid aluminum foil 100 nm thick, what intensity of x-rays can produce this in a single pulse? How does this intensity compare to the natural XFEL beam (parameters given above), if the beam is uniform in space and time.

   You may assume the pulse duration is much less than the average time for the electrons to recombine.

   (b) Such beams can be focussed, using special lenses. Well above the K-edge energy, the dielectric function of a metal may be represented as a free-electron gas, or plasma, having the function

   $$\frac{\epsilon}{\epsilon_\circ} = 1 - \frac{Ne^2}{m\epsilon_\circ\omega^2}\frac{1}{1+i\gamma/\omega},$$

   where N_e is the number density of electrons, m_e is the mass of the electron, ω is the angular frequency of the x-rays, γ is the collision rate for electrons, and you may assume $\gamma/\omega \ll 1.$

   If one makes from beryllium a spherical-surface lens 400 μm in diameter, what is the shortest focal length it will have, for 5 keV photons? Give details of the shape of your lens.

   If multiple such lenses can be used, how many lenses used together make a focal length of 1 m?

   (c) What is the fundamental physical limit to the smallest size of focal spot that might be formed?

   <u>Potentially useful information:</u>

   Some properties of Al: K-edge energy = 1.5596 keV; density ρ = 2.70 g/cm^3; mass of one Al atom = 4.48×10^{-23} g; mass absorption coefficient (at 1.5596 keV) $\mu/\rho = 3.621 \times 10^2$ cm^2/g.

   Some properties of Be: K-edge energy = 0.115 keV; density ρ = 1.85 g/cm^3; mass of one Be atom = 1.50×10^{-23} g; mass absorption coefficient (at 5 keV) $\mu/\rho = 4.369 \times 10^0$ cm^2/g.

9. A 10 kg sphere of ^{60}Co is moving at a velocity of 100 m/s in intergalactic space, according to an inertial observer. Cobalt-60 is radioactive, with a half-life $t_{1/2} = 1.6 \times 10^8$ s, and each nuclear decay emits gamma rays with an energy of 2.5 MeV. According to the observer, the gamma rays have a Doppler shift that depends on the direction in which they are emitted.

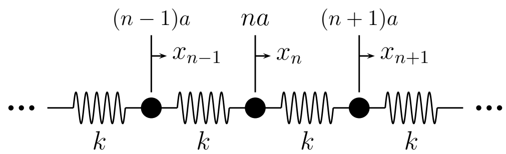
   (a) Show that the net force exerted by the gamma rays on the sphere is

   $$\vec{F} = -\frac{\dot{N}\hbar\omega}{c^2}\vec{v},$$

   where $\dot{N}$ is the number of decays per second from the Co nuclei in the rest frame of the sphere, $\hbar\omega$ is the energy of the gamma rays, and $\vec{v}$ is the velocity of the Co sphere.

   (b) Show that, despite this force, the sphere does not slow down.

10. A toy model of lattice vibrations (i.e. phonons) in a crystalline solid treats the crystal as an infinite linear chain of atoms in which each atom interacts only with its nearest neighbour atom, via a linear spring of spring constant k, as shown below. The equilibrium position of the the n^{th} atom in the chain is na, where a is the lattice constant, and x_n is the displacement from this equilibrium position, along the chain direction. Each atom has the same mass, m.



(a) Write the equation of motion of the n^{th} atom, and show that the following expression is a solution:

$$x_n(t) = A\mathrm{e}^{i[qna-\omega(q)\,t]},$$

and thus obtain the theoretical dispersion relation, $\omega(q)$ (i.e. wave frequency as a function of wave-vector q).

(b) Below are pictured experimentally determined phonon dispersion curves for two three-dimensional solids. The horizontal axis is the phonon wave-vector along various directions in q-space. Γ in the right-hand plot is at the origin. Based on these phonon dispersion curves, say as much as you can about the two solids, comparing their properties.