2018 Lloyd G. Elliott University Prize Exam

Compiled by the Department of Physics, University of Toronto

Tuesday, 13 March, 2018 | Duration: 3 hours.

Last name: ______________________ First name: ______________________

Institution: ______________________________________

Instructions:

1. You have 3 hours to complete the exam.
2. A non-programmable calculator is permitted but textbooks or reference materials are not.
3. There are 10 questions, not all of equal difficulty, but they all carry equal weight. It is unlikely that you will be able to answer every question, therefore you should budget your time wisely.
4. Write your solutions on the pages provided. Use the back of the pages if more space is needed.

Fundamental constants:

$e = 1.60 \times 10^{-19}$ C

$c = 3.0 \times 10^{8}$ m/s

$h = 6.63 \times 10^{-34}$ Js

$\hbar = 1.06 \times 10^{-34}$ Js

$\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$\mu_0 = 1.257 \times 10^{-6}$ H/m

$N_A = 6.02 \times 10^{23}$

$k_B = 1.38 \times 10^{-23}$ J/K

$R = 8.314$ J/K·mol

$m_p = 1.673 \times 10^{-27}$ kg

$m_e = 9.11 \times 10^{-31}$ kg

$\mu_B = 9.274 \times 10^{-24}$ J/T

$\mu_N = 5.05 \times 10^{-27}$ J/T

$a_0 = 0.529 \times 10^{-10}$ m

$1\ Ry = 13.606$ eV

1. The highest-energy cosmic rays are protons with energies of several Joules. Consider a 10 J cosmic ray proton that is being 'chased' across the Milky Way Galaxy by a photon.

   At time $t_0 = 0$ s the proton enters the Milky Way galaxy. At time $t_1 = 10^{-9}$ s the photon enters at the same point, travelling in the same direction.

   (a) Calculate the times of the following two events, in the rest frame of the galaxy: (i) the cosmic ray proton exits the Milky Way galaxy (take the galaxy diameter to be 10^{21} m); (ii) the photon catches up to the cosmic ray.

   (b) Describe this problem from the viewpoint of an observer in the rest-frame of the proton, and give the times of these events in the proton's rest frame.

2. The following is a simple model of the flow of air in a tornado.

   Consider a fluid of constant mass per unit volume ρ, in a cylindrical geometry (r, θ, z) around $\hat{z}$ such that $\vec{v} = v_\theta(r)\hat{\theta}$. The 'vorticity', defined as $\vec{\omega}(r) = \vec{\nabla} \times \vec{v}$, is given as $\vec{\omega}(r) = \omega_0 \hat{z}$ (ω_0 is a constant) for $r \leq R$, and $\vec{\omega} = 0$ for $r > R$.

   (a) Use Stokes' theorem (which may be familiar from electromagnetism) on a horizontal circle of radius r to compute and then plot $v_\theta(r)$ in terms of $\omega_\circ$ and R.

   (b) The flow of incompressible (constant ρ) fluids is often expressed in terms of a potential function ϕ where $\vec{v} = \vec{\nabla}\phi$. Solve for ϕ in the region $r > R$.

   Note: The gradient of a scalar field $A(r, \theta, z)$ in cylindrical coordinates is

$$\vec{\nabla} A = \left[\frac{\partial A}{\partial r}, \; \frac{1}{r}\frac{\partial A}{\partial \theta}, \; \frac{\partial A}{\partial z} \right]$$

3. Consider a long horizontal cylindrical shell, of length L and radius R, mounted horizontally on a frictionless axle, allowing it to rotate freely about its longitudinal axis with a moment of inertia I. It is made of an electrically insulating and non-magnetic material. A massless string attached to a vertically hanging mass m is then wound around the cylinder drum. The mass is released from rest at time $t = 0$.

   (a) Determine the angular acceleration and kinetic energy of the system after the hanging mass has fallen a distance h.

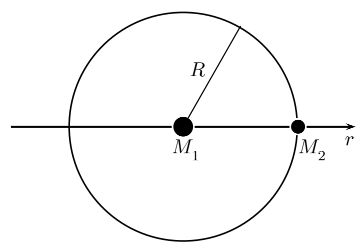
   (b) A net amount of positive charge Q, of negligible mass, is deposited uniformly on the outside drum of the shell before the mass is released. Redo part (a) under these conditions. Calculate the difference in kinetic energy between the two cases $Q = 0$ and $Q \neq 0$.

   Hint: The magnetic-induction field inside a very long solenoid of length L and N turns carrying a current I has a magnitude $B = \mu_0 NI/L$.

   (c) Explain as precisely as you can where the missing kinetic energy went.

4. The next-generation large space telescope, whose launch was originally planned for 2010, is now scheduled to be launched next year according to NASA. It will be put in orbit around the Sun, in a special zone where its distance relative to the Earth and to the Sun can remain constant. The location of such zones was first calculated in the XVIII$^{\text{th}}$ century by the French-Italian mathematician Joseph-Louis Lagrange.

   Even though it relies on approximations, Lagrange's full solution is fairly involved, but you should still be able to make a good qualitative guess at a partial solution. Consider two point masses, M_1 and M_2, referring respectively to the Sun and the Earth. Both orbit around their common centre of mass at angular velocity ω and with a period of one year. These orbits are circular to a good approximation, and the distance R between M_1 and M_2 is constant. Also, $M_1 \gg M_2$, allowing the centre of mass to be approximated as being at the centre of M_1.



   We wish to find where, on the line that joins M_1 and M_2, an object of mass m can sit so that it also orbits the centre of mass with the same, constant, angular velocity ω. We can safely assume that m is so small that it does not influence the motion of M_1 and M_2.

   (a) Write down the equation that must be satisfied by the forces acting on the orbiting m in terms of ω, M_1, M_2, R and r, where r is the distance between m and M_1, ie. the radius of its orbit around M_1.

   (b) Show that this condition can be written:

   $$u^3 - 1 \;=\; \pm \frac{\alpha\, u^2}{(1 \pm u)^2}$$

   where $u \equiv r/R$, and $\alpha \equiv M_2/M_1$.

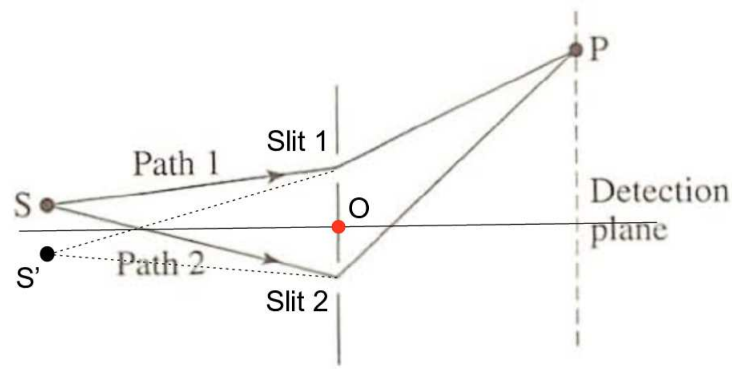
   (c) Do not attempt to solve this algebraic equation. Instead, use physical arguments to find how many solutions there are for r and where roughly the Lagrange zones are positioned on the r axis. Provide a qualitative sketch based on the above figure, and explain your reasoning.

   (d) The Webb space telescope (as it is called) will operate at a temperature of about 35 K and, therefore, it should be shielded from heat sources at all times, while having as unobstructed a view of the sky as possible. Briefly discuss which of your solutions is most suitable for the Webb telescope.

   Then, with the approximation $\alpha \ll 1$, estimate the distance from Earth where the telescope will be sitting.

   Hint: To a good approximation Kepler's Third Law for the system is $\omega^2 R^3 \;=\; G M_1$.

5. In 1920, Michelson first measured the angular size of a distant star, Betelgeuse in the constellation of Orion, using a 'stellar interferometer', illustrated below. Model the spatial extension of the star as two point sources of light, S and S', rather than a full disk. Consider S and S' to be independent sources, each emitting its own light that is not in phase with the other, thus light from S does not interfere with light from S'. Each source has its own set of interference fringes on the detection plane shown below. For convenience, however, assume that S and S' have the same color, with wavelength λ.



   (a) Denote the distance from S to Slit 1 by a, and distance from S to Slit 2 by b. (Analogous distances involving S' are obtained by symmetry.) What should be the difference $b - a$ be in order for the maxima in the interference pattern from S to overlap with the minima in the interference pattern from S'? (Hint: the answer is very simple and depends only on the wavelength.)

   (b) Denote the angle SOS' by α (this models the angular size of the star). It is extremely small. Express this angle in terms of the distance d between the slits and the difference $b - a$ that you found in (a). [Hint: S and S' are very far from the slits (astronomical distances, hundreds of light years).]

   (c) When Michelson observed Betelgeuse, at first he saw interference fringes. He then adjusted the distance between the slits so that interference minima and maxima produced by various parts of the star overlapped and resulted in a uniform image with no interference fringes. If we approximate the star by just our two point sources S and S', and assume that the wavelength is 600 nm and the distance between the slits is 3 meters when the fringes overlap, what is the angle SOS'?

6. According to Wikipedia, "*Cavitation is the formation of vapour cavities in a liquid — i.e. small liquid-free zones ('bubbles' or 'voids') — that are the consequence of forces acting upon the liquid. It usually occurs when a liquid is subjected to rapid changes of pressure that cause the formation of cavities where the pressure is relatively low. When subjected to higher pressure, the voids implode and can generate an intense shock wave.*" The shock waves are an important source of erosion for ship propellers.

   At time $t = 0$, a spherical bubble, whose centre is located at O and whose initial radius is a_i, forms in an infinite body of water. We neglect the influence of gravity, assume that the pressure inside the bubble is zero, and that its centre O is fixed in a Galilean reference frame. The evolution of its radius, $a(t)$, sets the fluid in motion, and a fluid velocity $\vec{v} = v(r,t)\hat{r}$ appears, with r the distance from O. Away from the bubble, the boundary conditions are $\vec{v} = 0$ and the pressure is $P = P_\infty$, constant in space and time. The flow is treated as incompressible, meaning that the mass per unit volume of the fluid, ρ, is constant.

   (a) i. If τ is the timescale over which the bubble collapses, apply dimensional analysis to the following formula
   $$\tau = k a_i^\alpha \rho^\beta P_\infty^\gamma \tag{1}$$
   to compute α, β and γ (make sure to use SI base units: m, kg, s). In this equation, k, α, β and γ are real, dimensionless numbers. (Recall that pressure has dimensions of energy per unit volume.)

   ii. Explain why the dependence of τ on a_i, ρ and P_∞ makes sense physically.

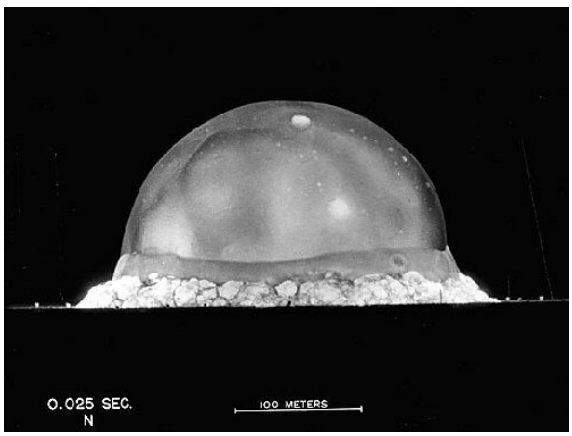
   (b) Because the flow is incompressible, the mass flux across a sphere of radius r, i.e., $Q = 4\pi r^2 \rho v(r,t)$ does not depend on r. Use the preceding statement, and the boundary conditions at the surface of the bubble, to show that
   $$v(r,t) = \frac{a^2(t)\dot{a}(t)}{r^2}, \tag{2}$$
   where $\dot{a} = \mathrm{d}a/\mathrm{d}t$.

   (c) From the equation above, it can be shown that that the radius of the bubble obeys
   $$a(t)\ddot{a}(t) + \frac{3}{2}(\dot{a})^2 = -\frac{P_\infty}{\rho}. \tag{3}$$
   You don't have to show this; rather, simply re-write this ODE with the following change of variables:
   $$A(t) = \frac{a(t)}{a_i} \qquad \text{and} \qquad T = \frac{t}{t_i}, \tag{4}$$
   and show that a smart choice for t_i makes the non-dimensional ODE universal (i.e. independent of the actual value of the physical parameters).

   (d) At time $t = t_1$ such that $a(t_1) = a_i/10$, the equation of motion takes the approximate form
   $$\frac{\partial}{\partial r}\left(\frac{P}{P_\infty}\right) \approx \frac{2}{15}\frac{a_i}{r^2}\left(\frac{a_i^3}{r^3} - 250\right). \tag{5}$$
   Show that pressure above has a maximum P_M for a certain distance r_M. Numerically compute r_M and P_M for $a_i = 1$ mm, $P_\infty = 10^5$ Pa and $\rho = 1000\ \mathrm{kg\,m^{-3}}$. Interpret the erosion of the propellers.

7. A recently published biography of Enrico Fermi recounts how he was able to estimate the yield of the first nuclear explosion (codename Trinity) in a New Mexico desert in July 1945. Standing $16\,\text{km}$ away from the site of the explosion, Fermi let scraps of paper fall and observed how far the air blast carried them. From this and the time elapsed when the air blast reached his position, he deduced that the yield was equivalent to 10 kilotons of TNT, the correct order of magnitude. Actually, Fermi's method only gave the energy released in the air blast; energy is also produced in the form of heat, light and radiation.

   For many years the official figure for the yield was kept secret by the US Army. In 1950, however, Geoffrey Taylor, a British physicist, obtained the yield of the test with a completely different and more reliable approach, using high-speed photographs published in Life magazine (one is shown below), from which the radius of the visible fireball at several times could be directly inferred.



   In this question, instead of the more famous "Fermi problem" first sketched above, you will atempt to reconstruct part of Taylor's argument.

   (a) Using dimensional analysis or otherwise, show that the dependence of the radius $R(t)$ of the fireball, as it expands, on the following quantities: total energy E released, time t, ambient air density ρ, and ambient air pressure P, can be written

$$R = k\left(\frac{Et^2}{\rho}\right)^{1/5},$$

   where k is a unitless constant. As it turns out, this dependence fits the data very well in the first $30\,\text{ms}$ or so of the explosion.

   (b) In the above still, $R = 140\,\text{m}$ at $t = 25$ ms. Air density at the site is $1.0 \times 10^3\,\text{kg/m}^3$. Also, 1 ton of TNT releases 4.2×10^9 J of energy. Assuming $k \simeq 1$, what is the yield in kilotons of TNT of the Trinity test predicted by your result of part (a)?

8. In this problem we use such units that $c = \hbar = 1$. We neglect masses of the electron, neutrinos, and antineutrinos.

   (a) A muon is a negatively charged particle, similar to the electron except that it is heavier and unstable. The Standard Model predicts that the muon decays into an electron, an electron anti-neutrino, and a muon neutrino, $\mu^- \longrightarrow e^- + \bar{\nu}_e + \nu_\mu$. Find the maximum energy of the electron in the muon rest frame, in terms of the muon mass m_μ.

   (b) Two experiments, Mu2e in Fermilab (USA) and COMET in J-PARC (Japan) are soon going to search for an exotic process of the conversion of a muon into an electron without any (anti)neutrinos. This process is predicted to occur in some extensions of the Standard Model. Due to energy and momentum conservation, this process is only possible when the muon decays near some other mass. Assume that the muon is bound to a nucleus of atomic number Z (the electric charge of the nucleus) forming a hydrogen-like atom. Assuming that the muon can decay into an electron and some virtual particle that is absorbed by the nucleus, and that the nucleus remains in its ground state, what is the energy of the produced electron in the initial atom's rest frame? Express your answer in terms of the muon mass m_μ, the mass of the nucleus m_N, the atomic number Z, and the fine-structure constant $\alpha \simeq 1/137$. [Treat m_μ/m_N and $Z\alpha$ as small parameters and keep only their leading powers.] Is this energy larger or smaller than the value you found in (a) and (assuming $m_\mu \ll m_N$ and $Z\alpha \ll 1$) by approximately how much?

   (c) The background that can mimic the conversion signal is due to highest-energy electrons produced in the Standard Model decay of the bound muon. Since the decaying muon and/or the produced electron can transfer momentum to the nucleus by an exchange of a photon, the energy of ν_μ and $\bar{\nu}_e$ can be negligible and the electron energy can reach the value you found in (b). How does the probability of emitting such high-energy electrons scale with Z? [Assume that m_N is very large so you can consider the nucleus as completely stationary both before and after the decay.]

9. Two identical massive planets (with mass $M = 10^{26}$ kg, and charge $Q = +10^9$ C) move in a circular orbit (of radius $R = 10^{11}$ m) around their mutual center of mass. Electromagnetic radiation is emitted by the accelerated charges, and gravitational radiation is emitted by the accelerated masses. In both cases, to leading order, the radiation is emitted due to the time-varying quadrupole moment of the charge and mass distributions.

   (a) Given that the power radiated by an electric dipole D oscillating at a frequency ω is

   $$P_{\mathrm{d}ipole} = \frac{1}{6\pi\epsilon_0}\frac{D^2\omega^4}{c^3},$$

   write down an expression for the electromagnetic power radiated by the planets. [Hint: We are going to be interested only in the relative magnitude of the electromagnetic and gravitational radiation, so you may argue by dimensional analysis to obtain an equation for the power radiated by an oscillating quadrupole to within a dimensionless constant k. Also, the electric quadrupole moment for two identical point charges Q separated by a distance R is QR^2.]

   (b) Using the similarity between Newton's law of gravitation and Coulomb's law of electrostatics, write down an expression for the gravitational power radiated by the planets. How does the radiated power scale with the distance between the planets, R?

   (c) Estimate the ratio of the power radiated into electromagnetic waves, to the power radiated into gravitational waves.

10. A modern technique of manipulating very small objects, with sizes ranging from nanometers to several micrometers, is based on trapping the object in a laser beam. The technique is called optical tweezers. Optical tweezers have been used to trap dielectric spheres, viruses, bacteria, living cells, organelles, metal nanoparticles, and even strands of DNA.

    This is a three-dimensional problem.

    A vertical beam of an IR laser with wavelength $\lambda = 1000$ nm is focused by a microscope objective lens to a spot, called the trap, which is in a volume of distilled water with suspended spherical particles. The diameter of the spheres is 50 nm; the material of the spheres is an isotropic dielectric. The beam power is P, and the laser beam has a Gaussian profile. The average distance between any two nearest spheres in water is much greater than the diameter of a sphere. Because its size is small compared to the wavelength of the light, the dielectric particle in the field of the laser beam may be treated as an induced dipole, with induced dipole moment $\vec{p} = \alpha\vec{E}$, where α is the polarizability of the material; moreover, this means that it obeys the condition of Rayleigh scattering of the laser radiation. Ignore spherical aberrations of the objective lens.

    (a) For the setup described above, find the forces exerted by the electromagnetic field of the laser on a dielectric sphere located near the focus spot of the laser beam. [Hint: It may be helpful to consider the potential energy of an electric dipole $\vec{p}$ in an electric field E.]

    (b) Explain why the spherical particle will be trapped by the laser beam.

    (c) How will the force you found change for a metal sphere of the same size?

    (d) If the trapped sphere is displaced from the center of the trap and released, what is its subsequent motion? Explain.

    (e) Sketch a side view of the objective lens, the laser beam after passing the objective, the trap, a sphere located near the trap, but not yet trapped, force(s) on the sphere and a convenient system of coordinates for reference.