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UNIVERSITY OF WATERLOO

2016 Lloyd G. Elliott University Prize Exam

Compiled by the Department of Physics & Astronomy, University of Waterloo

Tuesday, February 2, 2016 — Duration: 3 hours.

Last name: ____________________ First name: ____________________

Institution: ________________________

Instructions:

1. You have 3 hours to complete this exam.
2. A scientific calculator is permitted but textbooks or reference materials are not.
3. There are 10 questions, each weighted equally. It is unlikely that you will be able to answer each question, therefore you should budget your time wisely.
4. Write your solutions on the pages provided. Use the back of the pages if more space is needed.

Fundamental and astronomical constants:

- Atomic mass unit: $u = 1.661 \times 10^{-27}\,\text{kg} = 931.5\ \text{MeV/c}^2$
- Boltzmann's constant: $k_B = 1.381 \times 10^{-23}\ \text{J/K} = 8.617 \times 10^{-5}\ \text{eV/K}$
- Coulomb's Constant: $k = 1/4\pi\epsilon_0 = 8.988 \times 10^9\ \text{Nm}^2/\text{C}^2$
- Electron Mass: $9.109 \times 10^{-31}\,\text{kg} = 5.110 \times 10^{-1}\,\text{MeV/c}^2$
- Proton Mass: $1.673 \times 10^{-27}\ \text{kg} = 938.2\ \text{MeV}/c^2$
- Electron Volt: $1\,\text{eV} = 1.602 \times 10^{-19}\,\text{J}$
- Elementary Charge: $e = 1.602 \times 10^{-19}\,\text{C}$
- Permittivity of Vacuum: $\epsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}$
- Permeability of Vacuum: $\mu_0 = 4\pi \times 10^{-7}\ \text{Hm}^{-1}$
- Planck's Constant: $\hbar = h/2\pi = 1.055 \times 10^{-34}\ \text{J}\cdot\text{s} = 6.586 \times 10^{-16}\ \text{eV}\cdot\text{s}$
- Gravitational constant: $G = 6.672 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}$
- Speed of Light in vacuum: $c = 2.998 \times 10^8\,\text{m/s}$
- Radius of the Sun: $R_\odot = 6.950 \times 10^5\ \text{km}$
- Mass of the sun: $M_{SUN} = 2.0 \times 10^{30}\ \text{kg}$
- Luminosity of the sun: $L_{SUN} = 3.8 \times 10^{26}\ \text{J s}^{-1}$
- Mass of the hydrogen atom: $m_H = 1.7 \times 10^{-27}\ \text{kg}$
- mean distance from Sun to Earth: 1 Astronomical Unit (A.U.)=1.49×10^{11} m

Problem 1 : Statistical Physics

Consider a sheet of doped two-dimensional graphene with the Fermi energy $\epsilon_F > 0$. Find, up to dimensionless constants, the electronic specific heat of the graphene sheet in two limits: $k_B T \gg \epsilon_F$ and $k_B T \ll \epsilon_F$. You may assume the electron dispersion is linear, $\epsilon(\mathbf{k}) = \hbar v_F |\mathbf{k}|$. Your answer can be in the form of a dimensionless integral times a function of the temperature.

Problem 2 : Special Relativity

Two point masses, each with rest mass $m/2$, are attached to the ends of a massless rigid bar. This dumbbell is made to rotate about its center such that each mass moves with constant speed u, as shown. Let E denote the relativistic energy of the rotating dumbbell.

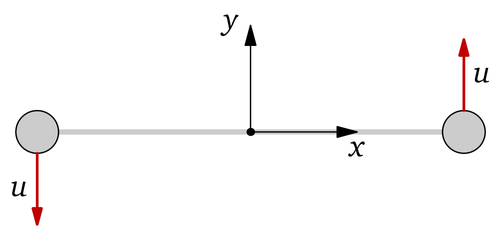
A What is E?

The center of the dumbbell is given a push in the x-direction so that its center is now moving with constant speed v to the right. Let E' denote the relativistic energy of the moving, rotating dumbbell.

B Consider the instant when the dumbbell is parallel to the x-axis. Starting with relativistic velocity addition, show that $E' = E/\sqrt{1 - v^2/c^2}$.

C Consider the instant when the dumbbell is parallel to the y-axis. Starting with relativistic velocity addition, show that E' is the same as in part **B** .

This illustrates that the energy associated with internal motions of parts of a system (e.g., quarks inside a proton) contributes to the rest mass of the system.

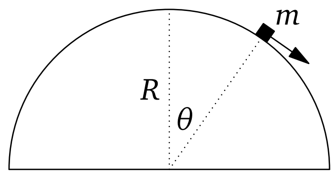


Problem 3 : Condensed Matter

Consider an undoped graphene sheet with the Fermi energy $\epsilon_F = 0$. Using the Drude model, which treats electrons as an ideal gas of classical particles experiencing impurity scattering with rate $1/\tau$ as they drift in an electric field, find the conductivity of this system as a function of temperature, up to dimensionless prefactors. You may take the electron dispersion to be linear $\epsilon(\mathbf{k}) = \hbar v_F |\mathbf{k}|$ and assume the impurity scattering rate $1/\tau$ is temperature-independent.

Problem 4 : Classical Mechanics

A student of mass m is sitting on top of a hemispherical igloo of radius R. The only forces at play are gravity, the normal force and friction. We assume that the coefficient μ of kinetic friction is the same as the coefficient of static friction.



A Find the smallest angle θ_0 from which the student will start to slide if released from rest. This will be the initial configuration for the remainder of the question.

B Derive an integral equation for the normal force $N(\theta)$ as a function of angle. ***Hint:*** *Take into account the work done by friction from θ_0 to the instantaneous position.*

C Solve this integral equation (or take its derivative with respect to θ and solve the resulting differential equation) by assuming a solution of the form

$$N(\theta) = A\cos\theta + B\sin\theta + Ce^{2\mu\theta},$$

where the constants A, B, C are to be found.

D Find a transcendental equation for the critical angle θ_c at which the student leaves the igloo's surface.

Problem 5 : Particle Physics

Particle A collides into particle B, which is at rest, and three or more particles are produced as a result: $A + B \longrightarrow C_1 + C_2 + \cdots + C_N$.

A Find the minimum energy that particle A must have for this reaction to take place in terms of the rest masses of the particles (m_A, m_B, m_1, m_2, etc.). This is called the *threshold energy*.

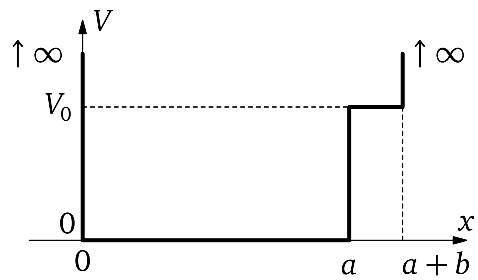
B Find the threshold energies for the reactions $p + p \to p + p + \pi^+ + \pi^-$ and $p + p \to p + p + p + \bar{p}$. The mass of the proton is 938 MeV and the mass of the pion is 140 MeV.

Problem 6 : Quantum Mechanics

Consider a particle of mass m in a one-dimensional potential given by

$$V(x) = \begin{cases} \infty & \text{if } x < 0 \\ 0 & \text{if } 0 < x < a \\ V_0 & \text{if } a < x < a+b \\ \infty & \text{if } x > a+b, \end{cases}$$

with x the position coordinate, $a, b > 0$, and $V_0 > 0$ (see figure).



A Solve the Schrodinger equation for an energy E with $0 < E < V_0$. Show that the energy E satisfies a relation of the form

$$e^{-2ika} = f(k, \tilde{k}, b). \tag{1}$$

Determine the function f, which depends on $k = \sqrt{2mE}/\hbar$, $\tilde{k} = \sqrt{2m(V_0 - E)}/\hbar$, and b.
Hint: *Apply boundary conditions at $x = 0$ and $x = a + b$ as for the infinite well potential problem. Require continuity of the wavefunction and its derivative at $x = a$.*

B Consider the limit $b \to 0$. Derive from Eq. (1) an energy equation keeping terms up to first order in b. Show that this equation agrees with the energy equation for an infinite well of width $a + b$,

$$e^{-2ik(a+b)} = 1,$$

when also taken to first order in b.

C Treat the potential step (between $x = a$ and $x = a + b$) as a perturbation. Argue that the result at the previous point is consistent with first order perturbation theory, assuming that the perturbation

$$V_1(x) = \begin{cases} 0 & \text{if } x < a \\ V_0 & \text{if } a < x < a+b \\ 0 & \text{if } x > a+b \end{cases}$$

is applied to the problem of infinite well in the interval $0 < x < a+b$. Find an expression for the energy corrections to first order in V_0 in the form of an integral. Show that in the limit $b \to 0$ this is consistent with Part **B** .

Problem 7 : Astronomy and Astrophysics

A One can make a remarkably accurate estimate of the central temperature of the Sun with just a few assumptions: that the density of the Sun is a constant throughout its interior, that the Sun is made of pure hydrogen, and that the ideal gas law governs the relationship between pressure, density, and temperature. Use the fact that gravity must be balanced by pressure (otherwise the Sun would contract relatively quickly) to derive a relationship between pressure and density across a spherical shell at a particular r (distance) from the centre of the Sun (this will be a differential equation of the form $dp/dr = f(r)$ called the Equation of Hydrostatic Equilibrium). Then, assuming that the pressure at the surface of the Sun is zero, use this to estimate the central pressure of the Sun. Then, use the ideal gas law to estimate the central temperature.

B Estimate the flux in neutrinos reaching the Earth from the centre of the Sun. The Sun's luminosity is powered by fusion of hydrogen to helium and each helium nucleus produced in these reactions is less massive than the sum of the masses of the hydrogen nuclei going into the reaction by 5.0×10^{-29} kg. Two neutrinos are produced for each helium atom created. Essentially none of these neutrinos are absorbed by the Sun and they propagate away at nearly the speed of light. What is the flux of these Solar neutrinos passing through the Earth?

Problem 8 : Electromagnetism

A Consider Maxwell's experimental test of Coulomb's law:

(i) Start with two concentric spherical conducting shells (outer radius a, inner radius b) insulated from one another, initially uncharged.

(ii) Charge the outer shell up to potential V_{test} with respect to infinity (ground).

(iii) Remove the source of potential, and put the inner and outer sphere into electrical contact.

(iv) Break contact between the shells, ground the outer shell, and measure the potential of the inner shell with respect to ground. Call this V_α.

Explain why V_α should be zero if Coulomb's law is exact.

B Consider a modified form of Coulomb's law, in which the field due to a point particle of charge q is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^{2+\alpha}}\hat{\mathbf{r}}. \qquad (2)$$

where $|\alpha| \ll 1$. With $\alpha \neq 0$ the value of V_α in Maxwell's test will no longer be zero. Derive the following relationship between a, b, V_{test}, α and V_α:

$$V_\alpha = \frac{\alpha V_{\text{test}}}{2}\left[n \ln\frac{n+1}{n-1} - \ln\frac{4n^2}{n^2-1}\right]$$

where $n = a/b$. This relation can be used to characterize possible deviations from Coulomb's Law, to first order in α.

Hint: *You can use the result that for the modified form of Coulomb's law (Eq. (2)), the field within a uniformly charged thin spherical shell with surface charge σ and radius R is (to first order in α):*

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0 (r/R)^2}\left[2(r/R) + \ln\left(\frac{1-(r/R)}{1+(r/R)}\right)\right]\alpha\,\hat{\mathbf{r}}$$

where r is the distance from the center of the sphere and $\hat{\mathbf{r}}$ is the unit vector pointing out from its center.

Also the following integral may be useful:

$$\int \frac{1}{x^2}\ln(1-x)\,dx = \frac{-1}{x}\ln(1-x) + \ln\frac{1-x}{x} + C.$$

Problem 9 : Waves

Find the analytic expressions for group velocity $v_g = v_g(\omega)$ and group index $n_g = n_g(\omega)$ for a dispersive material characterized by the relationship

$$k = \frac{\omega}{c}\sqrt{\left(\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}\right)} \qquad \text{where} \qquad \epsilon_2 = 1 - \frac{\omega_P^2}{\omega^2}$$

and ω_P is the Plasma frequency for a free-electron in a metal (you may set ϵ_1 constant). What is the effective wavelength for ω and what happens for $\omega \to \omega_P/\sqrt{1+\epsilon_1}$?

Problem 10 : Optics

Consider the situation illustrated in the figure, where a light ray propagating inside a prism undergoes total internal reflection. From Snell's law, we learn that if the incident angle θ_i of an incoming ray to a dielectric-air interface is greater than its critical angle $\theta_c = \arcsin(n_t/n_i)$, then total internal reflection occurs and all energy is reflected from the surface (See the Figure below), where $n_i(n_t)$ is the refractive index of the dielectric (air) with $n_i > n_t = 1$.

Let us examine if there really is no electromagnetic energy left in the air part of the region but in the close vicinity to the dielectric-air interface. The transmitted field will have a momentum specified by $\mathbf{k} = \mathbf{k}_{\parallel} \sin\theta_t + \mathbf{k}_{\perp} \cos\theta_t$, where $\parallel$ ($\perp$) refers to the field component along (perpendicular) to the dielectric air interface. Deduce that for a plane wave $\mathbf{E} = \mathbf{E}_0 \exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, there is a finite electromagnetic field which penetrates the interface: $E \propto e^{-\kappa z}$, with $\kappa = \omega/c\sqrt{(n_i \sin\theta_i)^2 - 1}$ and where z is the distance measured from the interface.

Hint: *Use the relationship* $\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = i\sqrt{\sin^2\theta_t - 1}$ *and Snell's law* $\sin\theta_t = n_i \sin\theta_i > 1$.